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# Aspects of the Two-Layer Model for Direct Contact Condensation of Steam on Wavy Falling Films

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Detailed physical modeling of the direct contact heat transfer process of steam condensing on a falling liquid film is a very difficult task due to the complex hydrodynamics of the film. The present state of the art is restricted to film Reynolds number of the order of 100. On the other hand, empirical relations cannot offer any insight into the mechanism and features of heat transfer in liquid film. Phenomenological models are needed to bridge the gap between empirical relations and direct physical simulations. One of these models is the so-called two-layer model, which divides the falling film into a laminar conduction-dominated (substrate) layer flowing over the solid wall and a completely mixed layer representing the waves. This model is further developed here by focusing specifically on two aspects. First, the influence of temperature-dependent physical properties of the liquid on the structure and heat transfer characteristics of the substrate layer is studied, and typical results are presented in the absence of waves. Second, the relation of the main parameter of the model (i.e., thickness of the substrate layer) to local film state is discussed in detail. Generalized constitutive laws and an approach based on utilization of the experimental film thickness time series are proposed and discussed. The proposed procedures can be integrated to a generalized two-layer model for direct contact condensation.

**Keywords:** Direct contact condensation; Falling films; Interfacial waves; Two-layer model

## Introduction

Transport processes in a falling liquid film constitute a very interesting problem, with several applications in chemical engineering. Most known applications are related to gas absorption by falling films (mass transfer in the film) (Killion and Garimella, 2001) and either direct contact condensation to or evaporation from falling films (heat transfer in the film) (Kreith and Boehm, 1988). Direct contact condensation of vapor on water falling films is of particular importance in regard to nuclear reactor safety (de la Rosa et al., 2009). It is noted that although many studies consider the two problems of mass and heat transfer as equivalent and extrapolate the results of one to the other, actually this is not the case. The mass transfer problem refers to the transport of a passive scalar in the film since the gas concentration is so low that it has no influence on liquid properties. On the contrary, during the heat transfer process the local temperature of the liquid changes and this, in turn, changes the physical properties of the liquid. In particular, the viscosity of water is very sensitive to temperature, resulting in a two-way coupling between heat transfer and film hydrodynamics. The focus of the present work is on vapor condensation on water falling films, but the proposed modeling framework can also be used for the mass transfer problem.

Modeling the condensation of vapor from air/vapor mixtures on flowing liquid films (direct contact heat transfer) is a very difficult problem and has been the target of extensive research. Detailed understanding of film structure and the corresponding transport processes is of paramount importance for overall condensation efficiency. Several approaches to tackling this particular problem can be found in the literature. The first of these is the derivation of empirical correlations for the overall condensation rate (e.g., for condensation in horizontal tubes: El Hajal et al., 2005; Thome et al., 2003; for gas absorption from falling films: Henstock and Hanratty, 1979; Won and Mills, 1982; Yoshimura et al., 1996). Better understanding of the process requires modeling of the evolution of condensation rate along the flow. In the case of the presence of non-condensables in the gas phase, modeling is separated into two parts: modeling of heat transfer in the flowing liquid and modeling of heat and mass transfer in the gas phase. The two problems are related through the appropriate interfacial conditions. Both problems would previously have been analyzed in terms of standard techniques from boundary layer approximation to the use of computational fluid dynamics codes for solving complete Navier–Stokes equations (or the corresponding Reynolds-averaged Navier–Stokes equations in the case of turbulent flow) if there had not been a major obstacle in the analysis (i.e., the transient wavy interface of the film). In the case of laminar flow of the film, several attempts have been made to solve the transport equations in both film and gas using a prescribed shape

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of the gas–liquid interface (Yang and Jou, 1993) based on either experimental observations (Jayanti and Hewitt, 1997a, b) or theoretical analysis of the fluid dynamics problem (Sisoev et al., 2005; Bontozoglou, 1998). The success of this approach is limited since it cannot reveal the complete picture but only indicative results of qualitative significance. The chaotic transient structure of the wavy interface cannot be approximated in general by a few specific patterns. Recent attempts based on the exact solution of the fluid dynamics problem using the volume of fluid approach are restricted to relatively low Reynolds numbers (<500) (Kunugi and Kino, 2005; Kunugi et al., 2005; Dietze et al., 2008).

The situation is more complicated in the case of higher liquid flow rates for which the flow in the field is turbulent. In this case, there are no theories for reproduction of the evolution of the interface so only the prescribed wave shape approach in conjunction with a  $k$ - $\varepsilon$  turbulent model can be employed (Jayanti and Hewitt, 1997a, b; Ye et al., 2002). A more usual approach is to assume a film of uniform thickness incorporating the influence of waves in the transport properties of the two phases, liquid and gas (Revankar and Pollock, 2005). Regarding the liquid phase, there are several models available for turbulent transport properties distribution in a falling film (Mudawar and El-Masri, 1986; Ueda et al., 1977). The major disadvantage of these models is that they predict a radical reduction of turbulent mixing at the imaginary surface of the film, posing a significant barrier to heat transfer. Regarding the gas phase, it is well known experimentally that a wavy interface increases the effective transfer coefficients (de la Rosa et al., 2009). This may be due to the combined effect of roughness and transverse velocity flow field imposed on the gas by wave motion.

A different modeling approach to the above condensation process was proposed by Karapantsios et al. (1995). Based on simulation results for flow field, it was found that recirculation patterns exist within the waves. This implies that convective currents transfer fast heat to the inner layers of the film (which is completely different from the interface mixing reduction proposed by turbulent models). Based on this observation and considering the low reliability of turbulent models, the following film model is proposed: The film consists of two regions – (i) the wave region where recirculation patterns ensure the transverse homogeneity of transported quantities and (ii) the laminar substrate region in which recirculation patterns have no influence and transverse transport occurs solely by conduction. The above model is henceforth termed the two-layer model. The substrate layer thickness used by Karapantsios et al. (1995) was assumed uniform along the flow and equal to the minimum measured film thickness. The relaxation of these two assumptions is the main objective of the present work, allowing the development of a generalized two-layer model for the direct contact condensation process.

The two-layer model will be examined here for the case of a planar falling film of cold water exposed to an infinite domain containing stagnant pure vapor. This choice was made in order to reduce the complexity of the problem and to restrict it to liquid phase transport. In particular,

some specific aspects of the two-layer approach are examined. The first of these concerns the effect of viscosity variation, due to its temperature dependence on the efficiency of condensation along a laminar falling film (or a laminar substrate layer). The second concerns the significance of the constitutive law for the thickness of the substrate layer on the results of the two-layer model. Finally, a framework is proposed for the derivation of the above constitutive law based on experimental time traces of film thickness.

## Main Part

### General

A practical question for falling film condensation is at what flow length is condensation complete: when all vapor is condensed. A water film is assumed falling over a vertical plate under fully developed flow entering a vapor chamber. This chamber is supposed to surround the vertical plate entirely, with a width many times the thickness of the flowing film such that the vapor distant to the film surface can be considered effectively stagnant and, in addition, the vapor pressure remains constant (not affected by vapor removal as condensate). The inlet mass flow rate of water per wetted width of the solid plate is  $G_o$ , the entrance temperature of water is  $T_o$ , and the temperature of vapor is  $T_s$  (only the value  $T_s = 100^\circ\text{C}$ , implying normal ambient pressure, will be considered here). The inlet water temperature,  $T_o$  is usually equal to ambient temperature and so lies between 10 and 30°C. The vertical plate is assumed to be externally insulated and is made of an insulating material. As feed water falls down the plate, vapor condenses on the film increasing its temperature until it reaches  $T_s$ . At this point, condensation stops since there no driving force (temperature difference) remains. It can be assumed that in the temperature range of interest ( $T_o$ – $T_s$ ) the density,  $\rho$  and specific heat capacity,  $c_p$  of water are practically constant. From a very simple integral energy balance of the process one can determine that the maximum flow rate of the condensate is  $G_{\text{cond}} = c_p(T_s - T_o)G_o/\Delta H$  ( $\Delta H$  is the latent heat of condensation). This means that the maximum condensation efficiency (condensed vapor/used water) is  $\frac{G_{\text{cond}}}{G_o} = \frac{c_p(T_s - T_o)}{\Delta H}$ . So the efficiency of the process is generally poor (about 14%) and is easily computed. What flow length is needed in order to achieve the maximum efficiency calculated above?

The case of a laminar film with one-dimensional flow (the effect of waves is ignored) is first studied. This problem will help on the one hand to extract the length scales of the phenomena occurring and, on the other, to focus on the problem of viscosity variation with temperature (which is usually ignored). In the case of constant properties (viscosity  $\mu$  and thermal conductivity  $k$ ), film thickness  $\delta$  and velocity profile  $u(y)$  are derived from well-known equations (Faghri and Zhang, 2006):

$$\delta = \left( \frac{3\mu G_o}{\rho^2 g} \right)^{1/3} \quad (1)$$

$$u(y) = \frac{\rho g}{\mu} \left( \delta y - \frac{y^2}{2} \right) \quad (2)$$

The heat conservation equation in the film can be written in this case as

$$u(y) \frac{\partial T}{\partial x} = \alpha \frac{\partial^2 T}{\partial y^2} \quad (3)$$

where  $\alpha = k/\rho c_p$  is the thermal diffusivity of water.

Using Equations (1) and (2), it can be shown that with the following nondimensionalization  $\eta = y/\delta$ ,  $t_s = x\mu\alpha/(\rho g\delta^4)$ , and  $\bar{T} = (T-T_s)/(T_o-T_s)$ , Equation (3) takes the form

$$\left( \eta - \frac{\eta^2}{2} \right) \frac{\partial \bar{T}}{\partial t_s} = \frac{\partial^2 \bar{T}}{\partial \eta^2} \quad (4)$$

which must be solved with initial/boundary conditions  $\bar{T} = 1$  at  $t_s = 0$ ,  $\bar{T} = 0$  at  $\eta = 1$ , and  $\frac{\partial \bar{T}}{\partial \eta} = 0$  at  $\eta = 0$ . It is evident that the mathematical problem defined by Equation (4) with its boundary conditions does not depend on any parameter of the physical problem (i.e., it admits a “universal” solution). The physical problem involves the length scaling according to which length needed for temperature equilibration (i.e., for completion of condensation) is proportional to length  $\frac{(3G_o)^{4/3}\mu^{1/3}}{g^{1/3}\rho^{5/3}\alpha}$ . The proportionality constant can be found from the solution of Equation (4). Condensation length increases as the mass flow rate of water increases, and also increases as viscosity increases despite the consequent film thickening according to Equation (1). Reduction in heat transfer due to film thickening is counterbalanced by decrease in velocity and increase in contact time as viscosity increases. Finally, increase in contact time predominates over the influence of film thickening. Whereas increase in  $\alpha$  from 20 to 100°C is about 15% and it may be ignored, the respective decrease in  $\mu$  is 72% leading to a characteristic length lower by 33% as temperature changes from  $T_o$  to  $T_s$ .

### Effect of Temperature Dependence on Viscosity

The above analysis showed that variation in viscosity variation with temperature is so large that it affects both the heat transfer process and the structure of the film (considerable thinning occurs along the flow as temperature increases and viscosity decreases). To make things even more complex, viscosity exhibits not only an axial but also a transverse profile due to the transverse velocity profile. This means that the velocity profile in Equation (2) derived for constant viscosity does not hold. A more detailed model incorporating both the temperature dependence of  $\mu$  and  $\alpha$  and increase in water flow rate due to condensation is next formulated. The x-momentum conservation equation is written as

$$\frac{\partial}{\partial y} \mu \frac{\partial u}{\partial y} = -\rho g \quad (5)$$

The viscosity is written as  $\mu(T) = \mu_o M(T)$  where  $\mu_o = \mu(T_o)$  [i.e.,  $M(T)$  is simply the normalized viscosity]. Equation (5) is solved using the no-slip boundary condition on the wall and the zero shear stress condition at the gas–liquid interface  $y = \delta$ , resulting in

$$u(y) = \frac{\rho g}{\mu_o} \int_0^y \frac{\delta - z}{M(T)} dz \quad (6)$$

The symbol  $z$  is used in the present work only as dummy variable for integrations.

Mass flow rate is given by integrating the velocity profile and interchanging the order of the double integral:

$$G = \rho \int_0^\delta u(z) dz = \frac{\rho^2 g}{\mu_o} \int_0^\delta \frac{(\delta - y)^2}{M(T)} dy \quad (7)$$

The above equation locally relates film thickness, mass flow rate, and the temperature profile. The heat transfer equation takes the form (transverse convection can be ignored for the thin films considered)

$$\frac{\partial u(y)T}{\partial x} = \frac{\partial}{\partial y} \alpha(T) \frac{\partial T}{\partial y} \quad (8)$$

The boundary conditions are similar to those of Equation (3), but now  $\delta$  has no constant value and evolves along the flow. Thermal equilibrium at the gas–liquid interface suggests that  $\dot{m}\Delta H = -k(T_s) \left( \frac{\partial T}{\partial y} \right)_{y=\delta}$  where  $\dot{m}$  is condensate flux at the film surface. Using this condition it can be shown that local mass flow rate, which is the sum of the initial and condensate flow rates, is given as

$$G = G_o + \frac{k(T_s)}{\Delta H} \int_0^x \left( \frac{\partial T}{\partial y} \right)_{y=\delta} dz \quad (9)$$

Inlet film thickness,  $\delta_o$  is related to inlet mass flow rate through

$$\delta_o = \left( \frac{3\mu_o G_o}{\rho^2 g} \right)^{1/3} \quad (10)$$

In order to render the problem in a more conceivable form, the following nondimensionalization is introduced:

$$\bar{x} = \frac{x}{L} \quad \bar{y} = \frac{y}{\delta_o} \quad \bar{\delta} = \frac{\delta}{\delta_o} \quad \bar{u} = \frac{u}{u_o} \quad A(T) = \frac{\alpha(T)}{\alpha_o} \quad (11)$$

where  $\alpha_o = \alpha(\tau_o)$  and  $u_o = \frac{\rho g \delta_o^2}{\mu_o}$ .  $L$  is an arbitrary length used to render  $x$  dimensionless. The scale value relevant to the practical problem,  $L = 1$  m is used here.

The nondimensional problem (the temperature is left in its dimensional form) is

$$C_1 \frac{\partial \bar{u}(\bar{y})T}{\partial \bar{x}} = \frac{\partial}{\partial \bar{y}} A(T) \frac{\partial T}{\partial \bar{y}} \quad (12)$$



$$\text{with } \bar{u}(\bar{y}) = \int_0^{\bar{y}} \frac{\bar{\delta} - z}{M(T)} dz \quad (13)$$

The initial/boundary conditions are

$$T(0, \bar{y}) = T_o, \quad T(\bar{x}, \bar{\delta}) = T_s, \quad \left( \frac{\partial T}{\partial \bar{y}} \right)_{\bar{y}=0} = 0 \quad (14)$$

The local value of film thickness must fulfill the following equation:

$$\bar{\delta} = \left( 3 \int_0^{\bar{\delta}} \frac{(\bar{\delta} - \bar{y})^2}{M(T)} d\bar{y} \right)^{-1/3} \left( 1 + C_2 \int_0^{\bar{x}} \left( \frac{\partial T}{\partial \bar{y}} \right)_{\bar{y}=\bar{\delta}} dz \right)^{1/3} \quad (15)$$

where

$$C_1 = \frac{u_o \delta_o^2}{\alpha_o L} \quad (16a)$$

$$C_2 = \frac{k(T_s)L}{\Delta H G_o \delta_o} \quad (16b)$$

In the above equation,  $C_1$  defines the length scale for condensation and  $C_2$  denotes the contribution of the condensate to film thickness.

The system of Equations (12)–(14) constitutes a free boundary problem and its numerical solution is not straightforward. First, the boundary immobilization technique is used to transform the mathematical problem to a fixed boundary one. The new pair of independent variables,  $t = \bar{x}, \eta = \bar{y}/\bar{\delta}$  is considered. By using the following chain differentiation rule:

$$\frac{\partial \Lambda(x, y)}{\partial x} = \frac{\partial \Lambda(t, \eta)}{\partial t} + \frac{\partial \Lambda}{\partial \eta} \frac{d\eta}{dt} \quad (17)$$

Equation (12) after some algebra is transformed into

$$\frac{\partial T}{\partial t} = \frac{1}{C_1 \bar{u}(\eta) \bar{\delta}^2} \frac{\partial}{\partial \eta} A(T) \frac{\partial T}{\partial \eta} + \frac{\eta d\bar{\delta}}{\bar{\delta} dt} \frac{\partial \bar{u} T}{\partial \eta} - \frac{T}{\bar{u}(\eta)} \frac{\partial \bar{u}}{\partial t} \quad (18)$$

The boundary conditions now are defined according to  $\eta = 0$  and  $\eta = 1$ . The equation for dimensionless thickness takes the form

$$\bar{\delta}^2 = \left( 3 \int_0^1 \frac{(1 - \eta)^2}{M(T)} d\eta \right)^{-1/3} \left( 1 + C_2 \frac{1}{\bar{\delta}} \int_0^t \left( \frac{\partial T}{\partial \eta} \right)_{\eta=1} dz \right)^{1/3} \quad (19)$$

The new velocity can be written as  $\bar{u}(\eta) = \bar{\delta}^2 \int_0^{\eta} \frac{1-z}{M(T)} dz$ .

The boundary immobilization procedure simplified the structure of the problem at the cost of adding complexity to the main partial differential Equation (18). The first term

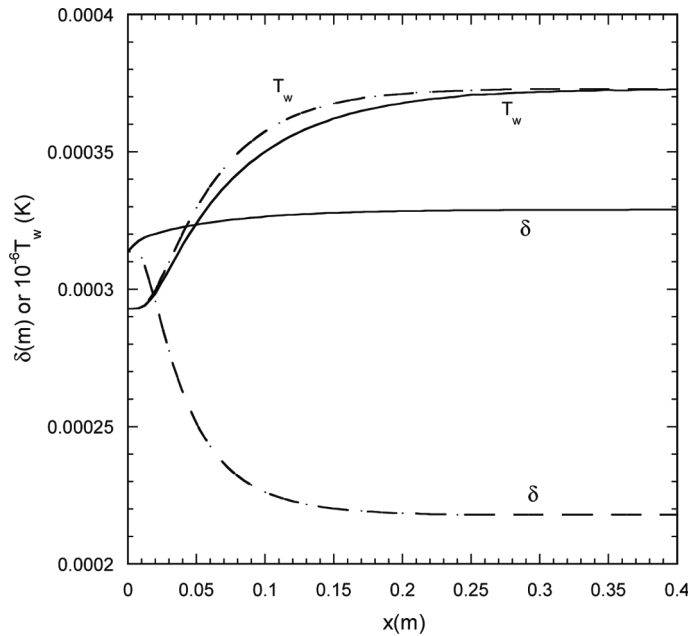
on the right-hand side is the usual conduction term, while the second term is a convection term induced by change in the coordinates system. The third term is due to velocity variation along the flow. At first glance, the second term might be omitted since the slope of the thickness is very small, but a more precise analysis shows that this is not the case since variation in thickness occurs at the same length scale with temperature evolution. Computational results show that the convection term has a contribution of a few percent (at most) to the solution.

The general structure of the problem is not usual, so a specialized numerical approach is used for its solution. The conduction term renders the problem very stiff, and therefore a stiff integrator must be used for its solution. The second and third terms on the right-hand side have a form incompatible with standard integrators so a specific approach is followed. The partial differential Equation (18) is discretized in the  $\eta$  direction using the finite difference technique. An additional ordinary differential equation is added to solve for the integral in Equation (19). Equation (19) is solved iteratively at each time step until convergence. The implicit Runge–Kutta integrator with adjustable time step (STIFF3; Villadsen and Michelsen, 1978) is used for the integration of the system of Ordinary Differential Equations (ODEs). The time derivatives appearing on the right-hand side of the system of ODEs – derived from the second and third terms on the right-hand side of Equation (18) – are handled in a special way. With each attempt of the integrator to find a new step, these terms are computed using a first-order time finite-difference approach based on the current values of the unknown quantities and the values at the previous successive step of the integrator.

### Indicative Results and Discussion

Indicative results are presented next to clarify the effect of viscosity variation on falling film heat transfer. In all cases,  $T_o = 20^\circ\text{C}$  (a typical ambient water temperature) and  $T_s = 100^\circ\text{C}$  (condensation temperature at atmospheric pressure). The evolution of film thickness along the flow for  $G_o = 0.05 \text{ kg/m/s}$  is shown in Figure 1. If the viscosity is assumed constant for its inlet value, then film thickness will increase along the flow due to condensation. Since the efficiency of condensation for steam condensation at 1 atm is about 14% and the thickness scales with mass flow to  $1/3$  power, it can be estimated that the increase in film thickness is about 5%. This is the scenario presented in Figure 1. The situation is completely different in the case of temperature-dependent viscosity, where film thickness decreases along the flow due to decrease in viscosity induced by temperature rise. This decrease cannot be counterbalanced by the small increase resulting from condensation, so decrease finally predominates in film thickness evolution.

Decrease in film thickness along the flow accelerates the heat transfer process leading to faster (i.e., requiring a smaller flow length) condensation. The degree of acceleration of heat transfer due to film thinning can be assessed by comparing the wall temperature ( $T_w$ ) evolution along the flow between the cases of constant and temperature-dependent



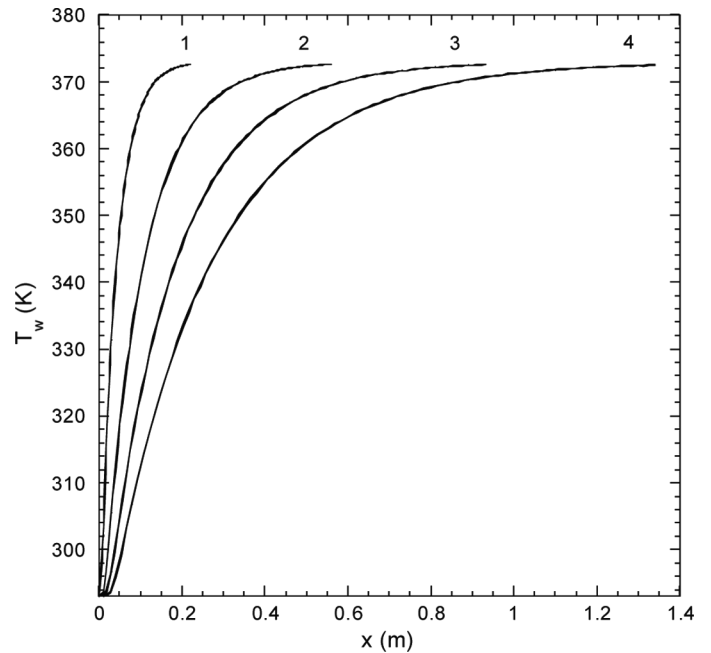
**Fig. 1.** Evolution of wall temperature and film thickness along the flow ( $G_o=0.05$  kg/m/s) for the cases of constant (solid lines) and temperature-dependent (dashed lines) viscosity.

viscosity (Figure 1). Despite the two curves arising together, at some point the temperature in the variable viscosity case increases more rapidly due to heat transfer enhancement induced by film thinning. The increase in heat transfer (and correspondingly, condensation) efficiency is not as dramatic as would be expected by the fourfold viscosity reduction (according to water property tables), and this is partly because heat transfer rate increases when the driving force (temperature difference) is already low.

Having clarified the importance of variable viscosity to overall condensation rate, the effect of mass flow rate of the liquid film is next examined. The evolution of wall temperature for four values of mass flow rate ( $G_o=0.05, 0.1, 0.15,$  and  $0.2$  kg/m/s) is shown in Figure 2. As expected, greater flow length is needed to complete condensation as water flow rate increases (leading also to increased condensate flow rate). The scaling of the required length to flow rate,  $G$  is close to the value of  $4/3$  previously estimated theoretically.

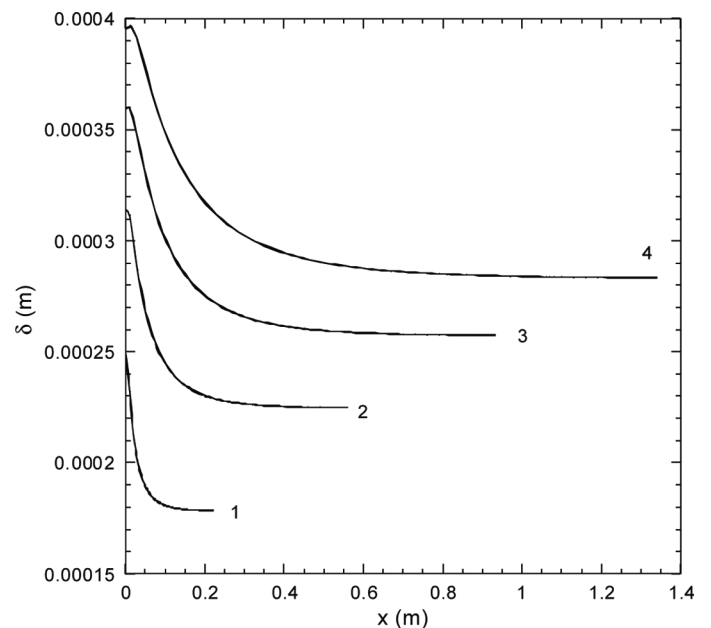
The evolution of film thickness for the four considered flow rates is presented in Figure 3. There is an initial region of rapid linear decrease in film thickness followed by a period of slow exponential decrease to an asymptotic value that corresponds to the thickness for viscosity value at temperature  $T_s$ .

It is of interest to examine the temperature and velocity profiles at several locations along the flow for some of the cases presented in previous Figures. The limiting cases of flow rates ( $G_o=0.05$  and  $G_o=0.2$  kg/m/s) are selected. The temperature profiles in the film for the large flow rate ( $G_o=0.2$  kg/m/s) at four locations along the flow are shown in Figure 4. There is a region close to the wall where the temperature profile is flat, with a value that increases along

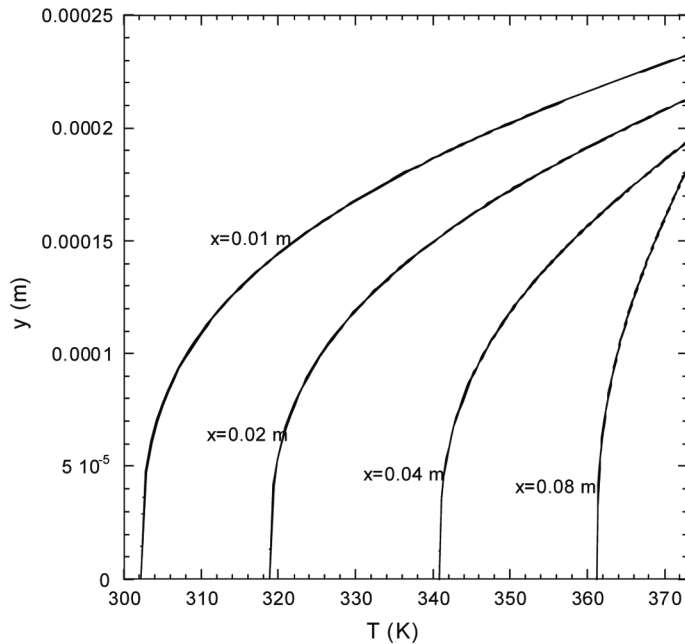


**Fig. 2.** Evolution of wall temperature along the flow for several values of mass flow rate  $G_o$ : curve 1,  $G_o=0.05$  kg/m/s; curve 2,  $G_o=0.1$  kg/m/s; curve 3,  $G_o=0.15$  kg/m/s; curve 4,  $G_o=0.2$  kg/m/s.

the flow; and there is a second region with a linear profile connecting the wall region of the (approximately) uniform temperature to the surface temperature,  $T_s$ . The corresponding velocity profiles are shown in Figure 5. The area between the curves and the vertical axis corresponds to the local flow

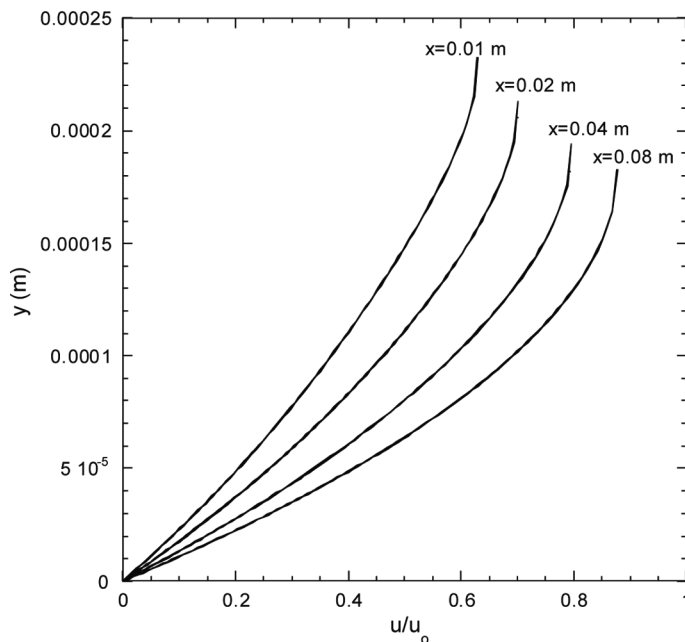


**Fig. 3.** Evolution of film thickness along the flow for several values of mass flow rate  $G_o$ : curve 1,  $G_o=0.05$  kg/m/s; curve 2,  $G_o=0.1$  kg/m/s; curve 3,  $G_o=0.15$  kg/m/s; curve 4,  $G_o=0.2$  kg/m/s.

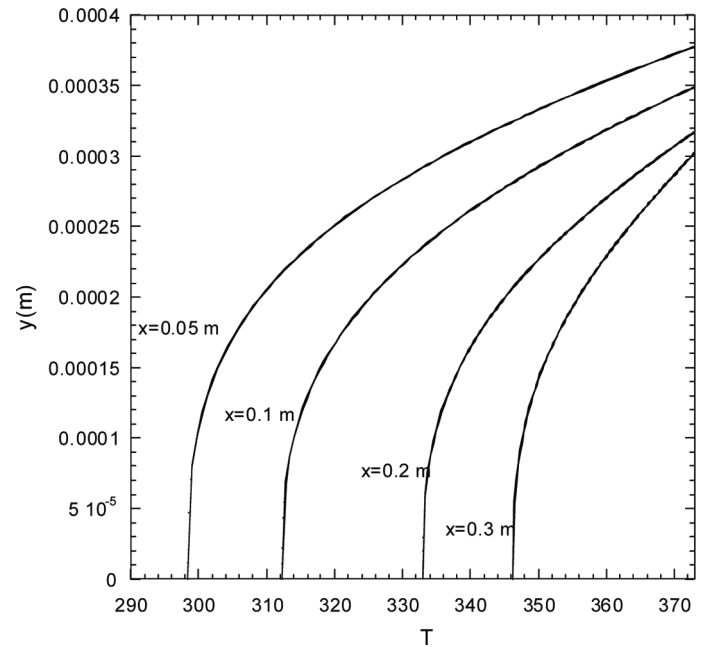


**Fig. 4.** Temperature profiles in the film at several locations along the flow ( $G_o = 0.05$  kg/m/s).

rate of the film. These areas increase for the curves in Figure 5 as  $x$  increases, due to the amount of condensate added to the film. According to Figure 4, as  $x$  increases the temperature variation across the film decreases and the velocity profiles approach that corresponding to uniform temperature (i.e., proportional to  $\eta - \eta^2/2$ ). As  $x$  decreases the velocity profile diverges from that of the uniform temperature, except at very small values of  $x$  where the

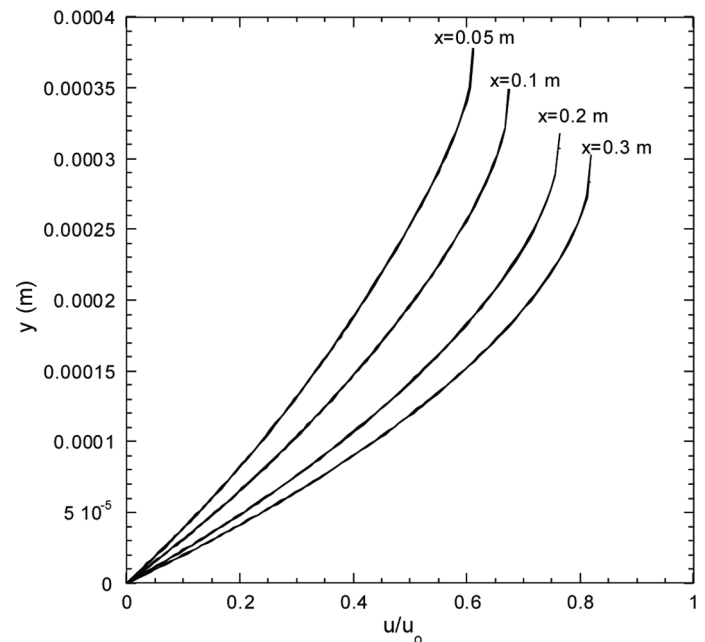


**Fig. 5.** Velocity profiles in the film at several locations along the flow ( $G_o = 0.05$  kg/m/s).



**Fig. 6.** Temperature profiles in the film at several locations along the flow ( $G_o = 0.2$  kg/m/s).

temperature profile is again uniform and the velocity profile is that for the uniform temperature. It is noted that although the velocity profile for the lower  $x$  value in Figure 5 is considerably different from the uniform temperature velocity profile, it can be perfectly fitted by a second-order polynomial curve. The temperature and velocity profiles for the higher ( $G_o = 0.2$  kg/m/s) flow rate case are shown in Figures 6 and 7, respectively. There is a qualitative similarity



**Fig. 7.** Velocity profiles in the film at several locations along the flow ( $G_o = 0.2$  kg/m/s).

between the low and high flow rates considered. However, the length scale for condensation completion differs between the two cases.

### The Two-Layer Model

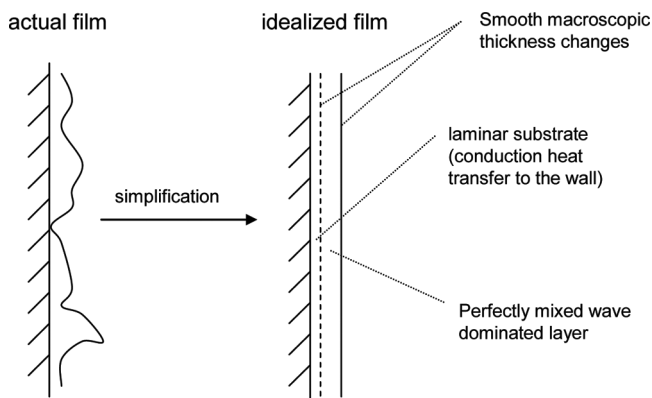
The two-layer model proposed by Karapantsios et al. (1995) is an alternative to the modified turbulent heat diffusivity expression, utilized for consideration of heat transfer enhancement induced by waves. The theory is that the film can split into two regions: The upper wave region is completely mixed so heat transfer occurs mainly through convection from the wave-induced flow field (see Figure 8). The region close to wall (substrate) exhibits laminar unidirectional flow and transverse heat transfer occurs via conduction.

In the two-layer model, the key parameter is the thickness,  $\delta_s$ , of the substrate film. In the case of uniform temperature this thickness can be assumed to be a function of Reynolds number; according to Karapantsios et al. (1995), it is assumed to take a constant value for all Reynolds numbers. The problem here is that as the temperature of the film increases due to condensation, time-averaged film thickness  $\delta_{ave}$  decreases along the flow so  $\delta_s$  must also decrease in order to avoid having  $\delta_s > \delta_{ave}$ . Thus the generalized constitutive law for  $\delta_s$  would take the form  $\delta_s = F(\text{Re}, \mu)$  where viscosity is computed at the local cross section average temperature. Reynolds number refers to the inlet condition (i.e., for  $\mu = \mu_o$ ). Viscosity is computed at the substrate-averaged (cup-mixing) temperature,  $T_{ave}$ . The degree of freedom is reduced by using the following decomposition to uncorrelated Re and viscosity dependence:

$$\delta_s = F_1(\text{Re})F_2(\mu) \quad (20)$$

Having established a constitutive law for thickness  $\delta_s$ , the heat transfer problem is formulated as

$$\frac{\partial u(y)\Gamma}{\partial x} = \frac{\partial}{\partial y} \alpha(T) \frac{\partial T}{\partial y} \quad \text{for } 0 \leq y \leq \delta_s \quad (21)$$



**Fig. 8.** A schematic of the two-layer approach for condensation of vapor on falling film.

with boundary conditions  $T(x, \delta_s) = T_s$  and  $\left(\frac{\partial T}{\partial y}\right)_{y=0}$  and the velocity profile given by Equation (6) with  $\delta_s$  substituted for  $\delta$ . The mass flow rate of the substrate region is computed from the relation

$$G_s = \frac{\rho^2 g \delta_s^3}{\mu_o} \int_0^1 \frac{(1-y^2)}{M(T)} dy \quad (22)$$

The wave layer is assumed to be in thermal equilibrium at the saturation temperature  $T_s$ , so the latent heat of the condensate should be equal to the heat transferred from the wave layer to the substrate layer, resulting in

$$G_{\text{cond}}(x) = -\frac{k(T_s)}{\Delta H} \int_0^x \left(\frac{\partial T}{\partial y}\right)_{y=\delta_s} dz \quad (23)$$

The mass flow rate of the wave layer at each position along the flow is derived from the following mass balance:

$$G_w = G_o + G_{\text{cond}} - G_s \quad (24)$$

The mathematical problem is closed (closure is achieved) using the constitutive law in Equation (20).

One shortcoming of the above model is that it assumes an instantaneous rise in the temperature of the wave region from  $T_o$  to  $T_s$ . To overcome this problem, finite (incomplete) mixing must be considered in the wave layer, which can be expressed as a heat transfer coefficient increasing the phenomenological parameters of the problem and adding a new equation for the wave layer temperature. In practice, condensation occurs not from pure vapor but from its mixing with a non-condensable gas (e.g., air). In that case, gas phase resistance is much higher than the effect of incomplete mixing so the wave layer can be assumed to be completely mixed and its temperature is dictated by gas phase heat and mass transfer. Overall, the two layers (wave and substrate) can be assumed as two heat tanks of differing capacity and thermal dynamics. The effect of substrate layer in the case of insulated external walls is delay (compared with the completely mixed film) in the condensation process, especially in the final stages. On the other hand, the two-layer model predicts faster condensation than the laminar flow model, which ignores the mixing effect of waves.

In order to present some results and to demonstrate the significance of various parameters, the simplest form of the constitutive law for  $\delta_s$  is introduced (the value  $\delta_{so}$  corresponds to inlet conditions):

$$\delta_s = \delta_{so} \left( \frac{\mu(T_{ave})}{\mu_o} \right)^a \quad (25)$$

In other words,  $\delta_s$  is not dependent on Re (compatible with the suggestion by Karapantsios et al., 1995) and has a power law dependence on viscosity. Exponent  $a$  is lower than  $1/3$  since the dependence of average film thickness

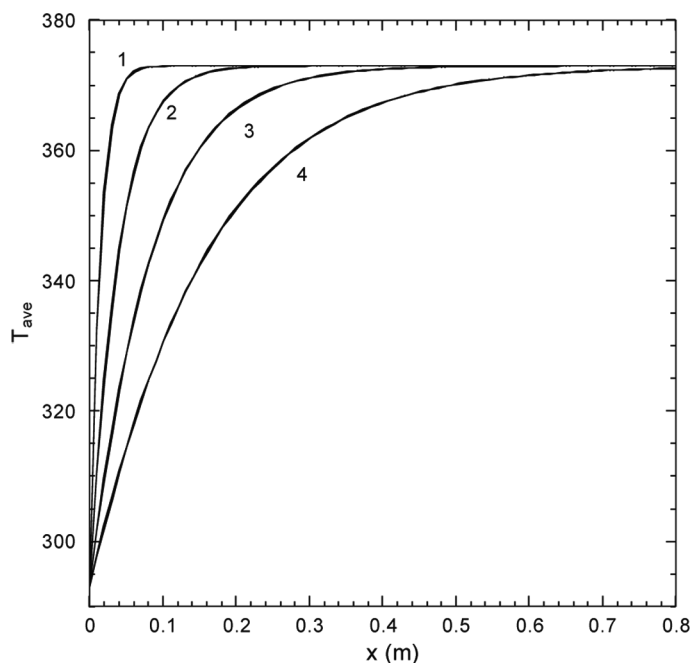


on the viscosity is of 1/3 power and sets the upper limit for the exponent in Equation (25). Equation (21) is solved using the same procedure as in Equation (12) so it is not repeated here. The only difference is that the thickness of the computation domain is computed from the constitutive law in Equation (25) rather than from the condition in Equation (15).

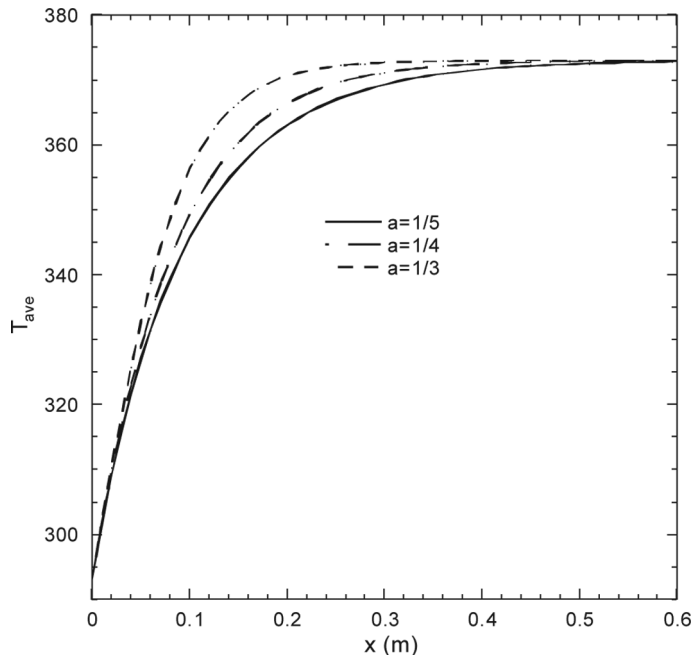
Evolution along the flow of the axial velocity-weighted average (cup-mixing) temperature of the substrate region ( $T_{ave}$ ), for several typical values of initial thickness  $\delta_{so}$  (defined by  $G_o$  and  $\mu_o$ ) and  $a=1/4$ , is shown in Figure 9. Heat transfer rate is a very strong function of  $\delta_{so}$  and increases as the latter decreases. This occurs because not only the thickness, but also the mass flow rate of the substrate layer, decreases as  $\delta_{so}$  decreases. The influence of the exponent  $a$  in the constitutive law on the evolution of substrate layer average temperature is shown in Figure 10 for  $\delta_{so}=0.3$  mm. Heat transfer rate increases and the condensation is completed more rapidly as exponent  $a$  increases, due to the faster thinning of the substrate layer according to Equation (25). From the above it is evident that the constitutive law dominates the heat transfer characteristics of the substrate layer.

#### An Approach for the Estimation of Substrate Layer Thickness from Film Thickness Experimental Data

In a previous work (Karapantsios et al., 1995) it was proposed that the minimum thickness of an experimental film thickness trace is a good candidate for substrate thickness. This minimum thickness shows a high degree of scattering from trace to trace, following rather the



**Fig. 9.** Evolution of average film substrate temperature for several values of initial thickness,  $\delta_{so}$ : curve 1,  $\delta_{so}=0.2$  mm; curve 2,  $\delta_{so}=0.25$  mm; curve 3,  $\delta_{so}=0.3$  mm; curve 4,  $\delta_{so}=0.35$  mm.



**Fig. 10.** Evolution of average film substrate temperature for several values of exponent  $a$  in the constitutive law for  $\delta_s$ .

statistics of rare events than regular statistics. An alternative method for substrate thickness estimation, taking into account the whole film thickness trace, must be developed. An example of such a theory and its application in the case of laminar films at a particular temperature, for which experimental information exists, is presented here.

In principle, computation of the velocity profile in a falling film requires solution of the complete Navier–Stokes equations or of some modified version thereof. Nevertheless, a rough estimation of velocity profiles at a fixed point in time can be made if the time trace of the local film thickness  $h(t)$  is known. The axial and transverse velocity of the liquid in the film ( $u$  and  $v$ , respectively) must obey the continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (26)$$

Since flow is laminar and viscosity is uniform, the small slope of film thickness can be invoked to apply the lubrication approximation which leads to the following quasi-parabolic velocity profile:

$$u(x, y, t_r) = 3u_{ave}(x, t_r) \left( \frac{y}{\delta} - \frac{1}{2} \left( \frac{y}{\delta} \right)^2 \right) \quad (27)$$

where  $u_{ave}$  is the average velocity at position  $x$ . It will be noted that this approximation is the leading-order expansion of the so-called long wave approximation theory, typically used in the study of falling films (Chang et al., 1996). Integrating Equation (26) with

respect to  $y$  and substituting Equation (27) leads, after some algebra, to

$$v = - \int_0^y \frac{\partial u}{\partial x} dy = -3 \left[ \frac{\partial u_{ave}}{\partial x} \delta \left( \frac{1}{2} \left( \frac{y}{\delta} \right)^2 - \frac{1}{6} \left( \frac{y}{\delta} \right)^3 \right) - \frac{\partial \delta}{\partial x} u_{ave} \left( \frac{1}{2} \left( \frac{y}{\delta} \right)^2 - \frac{1}{3} \left( \frac{y}{\delta} \right)^3 \right) \right] \quad (28)$$

Waves move with a celerity  $U$ , which is an experimentally measured quantity. This means that the spatial and temporal derivatives of film thickness and average velocity at a specific location are related by

$$\frac{\partial \delta}{\partial t_r} + U \frac{\partial \delta}{\partial x} = 0 \quad (29)$$

$$\frac{\partial u_{ave}}{\partial t_r} + U \frac{\partial u_{ave}}{\partial x} = 0 \quad (30)$$

The following local mass balance (accumulation of liquid must be compensated by increase in film thickness) must be valid at each  $x$  value:

$$\frac{\partial \delta}{\partial t_r} + \frac{\partial u_{ave} \delta}{\partial x} = 0 \quad (31)$$

Combining Equations (29) and (31) results in

$$\frac{\partial u_{ave}}{\partial x} = \frac{1}{\delta} (U - u_{ave}) \frac{\partial \delta}{\partial x} \quad (32)$$

Substituting Equations (32) and (29) into Equation (28) gives the following profile for  $v$ :

$$v = -3 \frac{1}{U} \frac{\partial \delta}{\partial t_r} \left[ (U - u_{ave}) \left( \frac{1}{2} \left( \frac{y}{\delta} \right)^2 - \frac{1}{6} \left( \frac{y}{\delta} \right)^3 \right) - u_{ave} \left( \frac{1}{2} \left( \frac{y}{\delta} \right)^2 - \frac{1}{3} \left( \frac{y}{\delta} \right)^3 \right) \right] \quad (33)$$

The only unknown in the above equation is the average velocity,  $u_{ave}$ , which can be computed by combining and integrating Equations (29)–(31):

$$(U - u_{ave})\delta = \left( U\delta_{ave} - \frac{G}{\rho} \right) \Rightarrow u_{ave} = U - \frac{U\delta_{ave} - G/\rho}{\delta} \quad (34)$$

The derivation of Equation (34) is not straightforward and its validity is proved in the Appendix.

By employing the experimental film thickness trace it is possible to estimate evolution of the transverse velocity profile at each particular location along the trace. For fully developed flow the velocity profiles must be statistically alike at every location. This transverse velocity in the presence of waves is very important because it is responsible for both mixing of liquid in the film and transferring heat through a convection mechanism instead of conduction, which is

the predominant transfer mechanism in the absence of waves.

In general, transverse velocity oscillates with time (along each wave), changing direction from toward the wall to toward the film surface. As an approximation, only the velocity toward the wall is important in regard to heat transfer (i.e., negative value of  $v$ ). Therefore, in order to derive a characteristic velocity profile for transverse heat transfer, the following time-averaged velocity is defined (still a function of distance  $y$  from the wall):

$$V(y) = - \frac{\int_0^P v(y, t_r) \Theta(-v(y, t_r)) \Theta(\delta(t_r) - y) dt_r}{\int_0^P \Theta(\delta(t_r) - y) dt_r} \quad (35)$$

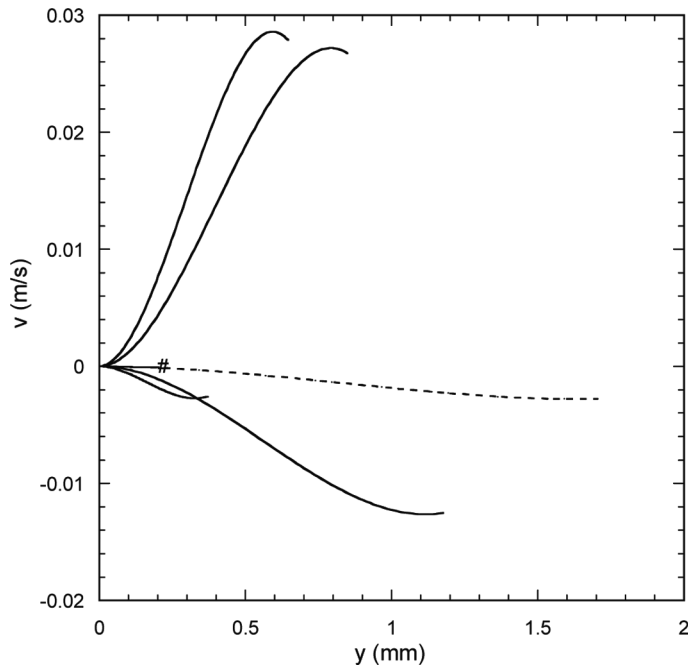
The function  $\Theta$  takes the value 1 when its argument is positive or 0 when it is negative. The equation above denotes that the averaging procedure holds only when film thickness exceeds  $y$ . The minus sign is used to render  $V$  positive. The value  $P$  in the integral is a time period sufficiently long for all the fluctuations to be absorbed and for convergence to a time-independent result.

Having defined a characteristic velocity toward the wall, the heat transfer problem can be considered as convection–conduction with convection decreasing close to the wall. In this context, a substrate with thickness  $\delta_s$  close to wall can be assumed in which conduction predominates. A Peclet number of this substrate can be defined based on its spatial average transverse velocity:

$$Pe(\delta_s) = \frac{V_{ave} \delta_s}{\alpha} = \frac{\int_0^{\delta_s} V(y) dy}{\alpha} \quad (36)$$

As  $\delta_s$  increases so does the  $Pe$  number computed from the above equation. For low  $Pe$  values, heat transfer in the film is conduction dominated. Although the transition from a conduction- to convection-dominated regime is smooth it can be approximated by a critical  $Pe$  number for transition (a typical choice for  $Pe_{cr}$  is between 1 and 3). The suggested value for  $\delta_s$  is the one that gives  $Pe(\delta_s) = Pe_{cr}$ . In this way, a theory is developed to estimate  $\delta_s$  based on experimental time series of film thickness and measured wave celerity.

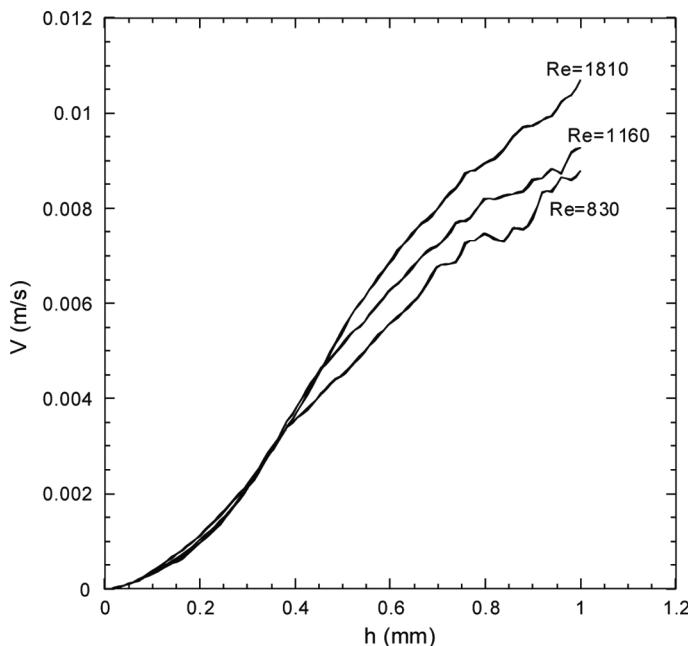
The above approach is applied in the case of falling water films at 20°C for three Reynolds numbers ( $Re = 830, 1160, \text{ and } 1810$ ) for which experimental data for  $h(t)$  and  $U$  exist (Kostoglou et al., 2010). Some instantaneous velocity  $v$  profiles for  $Re = 1810$  are shown in Figure 11. Velocity  $v$  is negative when its direction is toward the wall and positive in the opposite case. The profiles refer to different points in the time trace. The instantaneous film thickness at each instant corresponds to the highest value of  $y$  at each profile. Evidently, there is a variety of profiles with very different velocities. The symbol # corresponds to an instant of minimum thickness at which



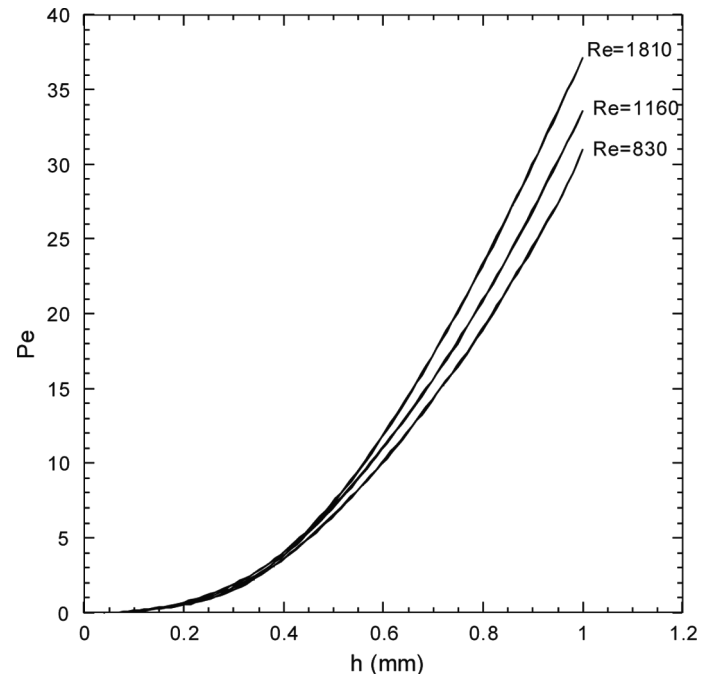
**Fig. 11.** Estimated snapshots of the transverse velocity profile  $v(y)$  based on experimental film thickness time series. # denotes minimum film thickness.

velocity is almost zero. The point here is that the average velocity may be sufficiently low to transfer heat for thicknesses larger than  $\delta_{\min}$ , so it is not necessary that  $\delta_{\min}$  and  $\delta_s$  coincide.

The velocity profiles  $V(y)$  for the three Reynolds given reach  $y=1$  mm and are shown in Figure 12. Fluctuation



**Fig. 12.** “Towards the wall” time-averaged velocity profile for three values of film Reynolds number (Re).



**Fig. 13.** Estimated Peclet number as a function of substrate thickness,  $\delta_s$  for three values of film Reynolds number (Re).

(noise) for lower Re number at high  $y$  values is due to the small portion of film thickness reaching this  $y$  value. It is apparent that velocity increases as Re increases. The corresponding profiles of the conduction substrate Pe number versus the thickness of the substrate are shown in Figure 13. It is interesting to observe that for the same thickness, Pe increases with Re due to increase in the wave effect. By assuming  $P_{cr}=2$  and using the profiles in Figure 13, the estimated values of  $\delta_s$  are 0.328 mm (Re = 830), 0.32 mm (Re = 1160), and 0.315 mm (Re = 1810). An initial observation is that the estimated values of  $\delta_s$  are much higher than the experimental values of  $\delta_{\min}$  ( $\sim 0.2$  mm) which is reasonable, as already explained. The second observation is that there is a (very slight) decrease in  $\delta_s$  with Re whereas  $\delta_{\min}$  increases (Karapantsios et al., 1995; Kostoglou et al., 2010). This decrease in  $\delta_s$  is due to the increase in wave power with mixing as Re increases.

The above approach can, in principle, be extended to the case of turbulent films using expressions for turbulent viscosity and turbulent thermal diffusivity, and by performing considerably more difficult computations. In order to be able to derive the constitutive law for  $\delta_s$  using this approach, experimental data for  $h(t)$  are needed for other temperatures between 20 and 100°C.

## Conclusions

In the present work we explored aspects of the two-layer model for the simulation of direct contact vapor condensation on falling films. In particular, the influence of the temperature dependence of liquid viscosity on velocity

profile in laminar falling film and, consequently, on the condensation process is examined. The significance of the constitutive law for substrate layer thickness is indicated and some simple forms for this law are presented and discussed. In addition, an approach is proposed for the derivation of this constitutive law based exclusively on film thickness experimental data. The analysis performed in this paper and the corresponding results constitute a major step toward the integration of the two-layer model for modeling the falling film direct contact condensation process.

### Nomenclature

$C_1, C_2$	dimensionless parameters defined in Equations (16a) and (16b)
$c_p$	water-specific heat capacity, J/kg/K
$g$	gravitational acceleration, m/s <sup>2</sup>
$G_{\text{cond}}$	condensate flow rate, kg/m/s
$G_o$	inlet flow rate, kg/m/s
$G_s$	substrate layer flow rate, kg/m/s
$k$	water thermal conductivity, J/s/m/K
$L$	characteristic length used only for nondimensionalization, m
$\dot{m}$	condensate flux, kg/m <sup>2</sup> /s
$M$	normalized viscosity, $\mu(\tau)/\mu_o$
$t$	normalized distance defined as $x/L$
$T$	water temperature, K
$T_o$	inlet water temperature, K
$t_r$	time, s
$t_s$	normalized distance along the flow, $x\mu\alpha/(\rho g\delta^4)$
$T_s$	condensation temperature, K
$u$	velocity component along the wall, m/s
$U$	wave celerity, m/s
$V$	time-averaged velocity toward the wall, m/s
$v$	velocity component normal to the wall, m/s
$x$	coordinate along the wall, m
$y$	coordinate normal to the wall, m
$z$	dummy integration variable

### Greek Characters

$\alpha$	thermal diffusivity of water, m <sup>2</sup> /s
$\alpha_o$	thermal diffusivity of water at inlet temperature, $T_o$ (m <sup>2</sup> /s)
$\Delta H$	phase change enthalpy of water, J/kg
$\delta_s$	substrate layer thickness, m
$\eta$	normalized distance from the wall, $y/\delta$
$\mu$	viscosity of water, kg/m/s
$\mu_o$	viscosity of water at inlet temperature, $T_o$ (kg/m/s)
$\rho$	density of water, kg/m <sup>3</sup>

### Subscripts and Superscripts

Overbar denotes dimensionless variables  
 Subscript “ave” denotes average value  
 Subscript “o” refers to inlet conditions

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## Appendix

### Proof of Equation (34)

Equation (34) denotes that the value  $(U - u_{ave})\delta$  is a constant. Taking the integral at all times, this constant can be found from the time-averaged quantities. It therefore remains to show that  $\frac{\partial(U - u_{ave})\delta}{\partial t_r} = 0$ , which can be written [utilizing Equations (29)–(31)] as

$$\begin{aligned} \frac{\partial(U - u_{ave})\delta}{\partial t_r} &= U \frac{\partial\delta}{\partial t_r} - \frac{\partial u_{ave}\delta}{\partial t_r} = \frac{\partial\delta}{\partial t_r} - \frac{1}{U} \frac{\partial u_{ave}\delta}{\partial t_r} \\ &= \frac{\partial\delta}{\partial t_r} + u_{ave} \frac{\partial\delta}{\partial x} + \delta \frac{\partial u_{ave}}{\partial x} = \frac{\partial\delta}{\partial t_r} + \frac{\partial u_{ave}\delta}{\partial x} = 0 \end{aligned}$$