

Self-similar growth of a gas bubble induced by localized heating: the effect of temperature-dependent transport properties

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Abstract

A gas bubble growing in a saturated liquid under the action of an internal heat source is studied theoretically as a model problem for the growth of a bubble around a miniature spherical heater submerged in the liquid. In general, this problem does not admit a self-similar solution but it is shown that for a heat source with strength increasing in proportion to the square root of time, a self-similar solution exists even under temperature-dependent transport properties. In that case the bubble radius is also proportional to the square root of time with a proportionality constant which can be found from the solution of a boundary value problem. The effect of temperature dependence of gas-in-liquid diffusivity and liquid and gas thermal conductivities on bubble growth is examined in detail. Finally, approximating explicit procedures are proposed for the computation of bubble growth rate without resorting to the solution of the boundary value problem.

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1. Introduction

The generation and growth of bubbles in liquids containing dissolved gases is a very important process in diverse scientific fields. Some of these are: materials technology, e.g. plastic foam manufacturing (Arefmanesh et al., 1990; Yoo and Han, 1982), human physiology, e.g. blood oxygenation, bubble growing in the tissue of airplane passengers, hypobaric and hyperbaric decompression of astronauts and divers, respectively (Kislyakov and Kopyltsov, 1988; Van Liew and Burkard, 1995; Srinivasan et al., 2000), geology, e.g. volcanic eruptions due to magma degassing (Prousevitch et al., 1983; Blower et al., 2001), food technology (Barker et al., 2002).

Typically, the bubbles are generated in the bulk of the liquid in large numbers making the visual observation of an isolated bubble a difficult task. To overcome this problem an experimental design has been proposed (Divinis et al., 2004) which admits the generation of just a single bubble in the liquid. This is realized by using a miniature thermistor as a heater, submerged in the bulk of the liquid. The thermistor is heated and as the liquid becomes locally supersaturated with respect to the dissolved gas, a bubble is generated. Subsequently, the bubble grows with transfer of mass of the dissolved gas from the bulk of the liquid to the bubble surface. The radially symmetric development of the bubble around the thermistor is highly desirable in order to make the process amenable to a rigorous mathematical treatment and so identify its features easier. The model problem for this experimental design is that of a radially expanding bubble with a heat source inside the bubble.

In general, bubble growth is a complex process involving combinations of mass, heat and momentum transfer

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between the expanding bubble and the liquid. There are a lot of studies regarding bubble growth dominated by heat transfer alone (vapor bubble, e.g. Robinson and Judd, 2001) and by mass transfer alone (gas bubble, e.g. Payvar, 1987). There is also some work reported on combined heat and mass transfer during vapor bubble growth from a binary solution with a non-volatile solute (Miyatake et al., 1994). It was not possible to find studies dealing with gas bubble growth induced by localized heating, which is also a process affected by coupled mass and heat transfer.

A common characteristic of all bubble growth problems is the radial flow induced to the liquid by the expanding bubble. This flow results in the appearance of convective terms, which change the character of the mass and heat conservation equations in the liquid and render their solution more cumbersome. Several asymptotic results have been developed for the growth under limiting conditions (Vrentas et al., 1983). For example, when the bubble grows slowly with respect to heat transfer kinetics then the temperature field can be assumed to be in a pseudo-steady state (Blower et al., 2001). On the opposite limit of a very fast growing bubble, a temperature boundary layer forms, which is thin with respect to bubble radius so its curvature can be ignored and analytic solutions or low order polynomial approximations of the temperature field are allowed (Plesset and Zwick, 1954; Rosner and Epstein, 1972).

An important class of solutions preserves the property of self-similarity, in which temperature and concentration profiles do not change with time when expressed in the radial coordinate normalized with the instantaneous bubble radius. As a result, the bubble grows under constant temperature and composition and its radius increases with the square root of time. Self-similar solutions have been reported for vapor bubble growth in a binary liquid (Scriven, 1959), multi-component gas bubble growth (Cable and Frade, 1987a) and gas bubble growth with concentration-dependent diffusivity (Lastochkin and Favelukis, 1998). All these cases correspond to a single growth mechanism, either heat or mass transfer.

The scope of the work is to find and parametrically investigate the self-similar solution for a gas bubble growing isothermally under the action of an internal heat source. The temperature dependence of the transport properties in the liquid and gas phase is taken into account in the analysis, and is treated both rigorously (by numerical computation) and approximately (by appropriate simplifications). The structure of the paper is the following: at first, the mathematical problem of bubble growth in a saturated liquid due to the existence of a heat source with varying strength inside the bubble is formulated. Then, it is shown that under certain conditions for the location and time dependence of the heat source, a self-similar solution for the growth problem is possible (isothermal bubble) and the corresponding equation in a self-similar variable is derived. The method of solution for the heat transfer part of the problem is discussed and an approximation is proposed which admits the replacement of the temperature-dependent liquid conductiv-

ity with an appropriate average. The complete self-similar problem is solved numerically and several results for the growth rate are derived. Finally, an approximate computation of the growth rate is proposed, which takes into account the effects of the diffusivity–temperature dependence and intrabubble temperature distribution without resorting to iterative procedures. Finally, comparisons between approximate and exact bubble growth rates are presented and discussed.

2. Problem formulation

The present work deals with relatively small bubble growth rates (of the order of mm/s) similar to those observed in the experiments of Divinis et al. (2004) performed in a microgravity environment. The equation of motion of the bubble (Rayleigh–Plesset equation) under these bubble growth conditions degenerates to

$$P_v + P_g = P_\infty, \quad (1)$$

where P_∞ is the ambient pressure, P_g is the partial pressure of the dissolved gas in the bubble and P_v is the vapor pressure of the solvent. The inertia and viscous terms which contain time derivatives of the bubble radius R can be ignored due to the slow growth of the bubble and also because only moderately viscous solvents are examined here. These terms are known to be important in the early stages of boiling applications, where typical growth rates are of the order of mm/ μ s. The contribution of surface tension to the bubble evolution has been studied in detail by Cable and Frade (1998). They found that for bubbles with radius 20 times larger than the critical one the effect of surface tension is negligible. The critical radius is of the order of 1 μ m, so the surface tension effect in Eq. (1) can be ignored for bubbles larger than 20 μ m (present case).

The equation of continuity in the liquid phase requires that the (radial) liquid velocity have the following form (Scriven, 1959) assuming that the density of the bubble is negligible with respect to the density of the liquid and ignoring any temperature dependence of the liquid density

$$u = \frac{R^2}{r^2} \frac{dR}{dt} = \frac{R^2}{r^2} \dot{R}. \quad (2)$$

Several time scales are imposed to the problem by the mass and thermal transport in both the liquid and the gas phase. The significance of the transient behavior for any one of the four combinations of transport phenomena (mass and heat transport in the liquid and gas phase, in pairs) depends on the ratio of their intrinsic rate over the rate of bubble growth. In principle, the mass and heat diffusivities in the gas phase are much larger than in the liquid phase. For bubble expansion rates less than 1 mm/s the bubble can be assumed to grow in a quasi-steady fashion as regards the gas phase. This means that the composition of the gas in the bubble can be taken as uniform. On the other hand, the quasi-steady temperature distribution is not uniform due to the existence

of the internal heat source. Convection inside the bubble can be safely ignored due to the low gravity conditions and the large thermal diffusivity of the gas. So, the temperature field inside the bubble $T_g(r, t)$ is given from the solution of

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left[r^2 k_g(T) \frac{\partial T_g}{\partial r} \right] = WH(r) \quad \text{for } 0 < r < R, \quad (3)$$

where W is the strength (power) of the internal heat source and k_g is the thermal conductivity of the gas phase. The function $H(r)$ stands for the spatial distribution of the heat source and must be such that at each moment

$$4\pi \int_0^R r^2 H(r) dr = 1. \quad (4)$$

The energy conservation equation in the liquid phase which must be solved for the temperature field $T(r, t)$ in the fluid is

$$\frac{\partial T}{\partial t} = \frac{1}{r^2} \frac{\partial}{\partial r} \left[r^2 k(T) \frac{\partial T}{\partial r} \right] - \dot{R} \frac{R^2}{r^2} \frac{\partial T}{\partial r} \quad \text{for } R < r < \infty, \quad (5)$$

where the first term is the energy accumulation, the second term is the conduction and the third term is the convection due to the expansion of the bubble. The density, specific thermal capacity and thermal conductivity of the liquid phase are denoted as ρ , c_p and k , respectively. In Eq. (5) the variation of the product ρc_p with temperature has been ignored, being in general much smaller than that of k .

In a similar way, the solute mass conservation equation in the liquid phase is written as

$$\frac{\partial c}{\partial t} = \frac{1}{r^2} \frac{\partial}{\partial r} \left[r^2 D(T) \frac{\partial c}{\partial r} \right] - \dot{R} \frac{R^2}{r^2} \frac{\partial c}{\partial r} \quad \text{for } R < r < \infty, \quad (6)$$

where c is the concentration and D the diffusivity of the gas in the liquid.

Since in most practical situations the mass fraction of the solute gas in the liquid phase is negligible (Patel, 1980) the mass balance of the gas on the bubble surface reads

$$\frac{d}{dt} \left(\frac{4}{3} \pi R^3 \rho_{gm} \right) = 4\pi R^2 D \left(\frac{\partial c}{\partial r} \right)_{r=R}. \quad (7)$$

The first term in this equation is the accumulation of the dissolved gas in the bubble and the second term is the diffusive flux of the gas from the liquid to the bubble integrated over the bubble surface. The mean molar density ρ_{gm} of the dissolved gas can be found by integrating the local molar density ρ_g across the bubble.

$$\rho_{gm} = \frac{3}{R^3} \int_0^R \rho_g r^2 dr. \quad (8)$$

The reason for the spatial variation of the gas density inside the bubble is the existence of the temperature distribution in the gas. The local molar density is related to the local

temperature through the ideal gas law

$$\rho_g = \frac{P_g}{R_g T_g}. \quad (9)$$

The boundary conditions on the gas–liquid interface are the continuity of the temperature

$$T(R, t) = T_g(R, t) = T_B(t) \quad (10)$$

and the equilibrium of the dissolved gas in the two phases

$$c(R, t) = c_{\text{sat}}(T_B), \quad (11)$$

where T_B is the temperature at the surface of the bubble and $c_{\text{sat}}(T)$ is the solubility of the gas in the liquid at temperature T and pressure P_∞ (total pressure) or equivalently, at partial pressure of gas P_g (Fogg and Gerrard, 1991).

Finally, the temperature dependence of the solvent vapor pressure is needed to close the problem.

$$P_v = P_{\text{sat}}(T_B). \quad (12)$$

The initial conditions of the problem are uniform initial temperature

$$T(r, 0) = T_g(r, 0) = T_0 \quad (13)$$

and uniform concentration of dissolved gas equal to the saturation concentration at temperature T_0

$$c(r, 0) = c_0 = c_{\text{sat}}(T_0). \quad (14a)$$

The initial size of the bubble must be the critical size, R_0 , i.e., the smaller size for which a thermodynamically stable bubble can exist.

$$R(0) = R_0. \quad (14b)$$

However, R_0 is a function of the supersaturation, which, for the present problem evolves with time as the temperature of the heating element ascends. Thus, it is not known in which supersaturation the bubble will form. Nevertheless, an exact value for R_0 is of little significance since, as with surface tension, very soon the bubble forgets its initial size.

The boundary conditions far from the bubble are

$$T(\infty, t) = T_0, \quad (15)$$

$$c(\infty, t) = c_0. \quad (16)$$

Finally, an important constraint is the heat balance for the bubble, which reads

$$W(t) = L(T_B) \frac{d}{dt} \left[\frac{4\pi}{3} R^3 \rho_{gm} (P_v/P_g) \right] + 4\pi R^2 P_\infty \dot{R} - 4\pi R^2 k(T_B) \left(\frac{\partial T}{\partial r} \right)_{r=R}. \quad (17)$$

The left-hand side of Eq. (17) is the rate of heat generation from the internal heat source, the first term on the right-hand side is the rate of energy consumed for the evaporation of

solvent (L is the phase change enthalpy at temperature T_B), the second term is the rate of gas expansion work (which will be omitted in what follows due to its small magnitude), and the third term is the rate of heat transfer from the liquid to the bubble. The terms corresponding to the energy of dissolved gas desorption, the change of the gas temperature, kinetic energy of the gas and surface tension work have been omitted since their contribution is negligible.

3. Existence of self-similarity

The above system of equations must be solved with the strength $W(t)$ of the heat source as an input, to give the evolution of the bubble radius R , the bubble surface temperature T_B and the temperature and concentration profiles in the liquid and gas phase at any time. The system consists of ordinary differential equations coupled with partial differential equations with moving boundaries, so in general it should be solved numerically.

As Divinis et al. (2004) experiments have shown, it is possible for the bubble to grow under quasi-isothermal conditions for a significant portion of the total growth period. For the special case of a constant bubble surface temperature, $T_B(t) = T_e$, a major simplification can be made by combining the two independent variables, x and t , and so reducing by one the dimensionality of the problem. In this case, the structure of the problem is changed. The input variable is no longer the source strength $W(t)$ but the constant temperature T_e . The system of equations must be solved now for a given value of T_e , to yield the bubble radius time-history, the temperature and concentration profiles and the heat source strength $W(t)$ which is required in order to retain the bubble surface temperature constant and equal to T_e .

The following non-dimensionalization is adopted for the case of a constant bubble surface temperature T_e to facilitate the solution of the problem:

$$\bar{T} = \frac{T - T_0}{T_e - T_0}, \quad \bar{k}(T) = \frac{k(T)}{k(T_0)},$$

$$\bar{D}(T) = \frac{D(T)}{D(T_0)}, \quad \tau = \frac{D(T_0)t}{R_0^2},$$

$$\bar{c} = \frac{c_0 - c}{c_0 - c_e}, \quad \bar{R} = \frac{R}{R_0},$$

$$\bar{r} = \frac{r}{r_0},$$

$$\bar{T}_g = \frac{T_g - T_e}{T_e}, \quad \bar{k}_g(T) = \frac{k_g(T)}{k_g(T_e)},$$

$$\bar{H}(r) = H(r)R_0^3, \quad \lambda = \sqrt{\frac{k(T_0)}{\rho c_p(T_0)D(T_0)}},$$

$$\bar{W} = \frac{W}{4\pi R_0 k(T_0)(T_e - T_0)},$$

where $c_e = c_{\text{sat}}(T_e)$. The parameter λ is the ratio of the heat to mass penetration depths in the liquid. Using the above non-dimensionalization the following dimensionless parameters result:

$$C_1 = \frac{D(T_0)\rho_{ve}L(T_e)}{k(T_0)(T_e - T_0)}, \quad (18)$$

$$C_2 = \frac{k(T_0)(T_e - T_0)}{k_g(T_e)T_e} \quad (19)$$

$$\text{and } F = \frac{c_0 - c_e}{\rho_{ge}} \quad (20)$$

is the so-called Foaming number. The parameter C_1 is related to the ratio of heat transfer due to the water evaporation to the conductive heat transfer from bubble to liquid.

The reference densities ρ_{ge} and ρ_{ve} are computed at the bubble surface temperature T_e using Eqs. (1) and (9).

The initial conditions take the form

$$\bar{T}(\bar{r}, 0) = \bar{c}(\bar{r}, 0) = 0, \quad (21a)$$

$$\bar{R}(0) = 1 \quad (21b)$$

and the boundary conditions

$$\bar{T}(\infty, \tau) = \bar{c}(\infty, \tau) = 0, \quad (22a)$$

$$\bar{T}(\bar{R}, \tau) = \bar{c}(\bar{R}, \tau) = 1, \quad (22b)$$

$$\bar{T}_g(\bar{R}, \tau) = 0. \quad (22c)$$

The equations describing the problem are of the transient convection–diffusion type in a semi-infinite domain with a moving boundary and their numerical solution is not trivial. A standard procedure is the immobilization of the moving boundary by using a new spatial variable $y^3 = r^3 - R^3$ (Lagrangian transformation). This choice has the advantage of eliminating the convection terms (Arefmanesh et al., 1992; Prousevitch et al., 1983). On the other hand, the required temperature and concentration profiles are extremely steep with respect to the new variable. This makes necessary a new (fixed in time) transformation of variable which has as a consequence the reappearance of the convection terms. So, the advantage of the above variable transformation vanishes. The alternative Lagrangian transformation $y = r/R$ is more meaningful from a physical point of view.

By using the following chain differentiation rule

$$\frac{\partial X(r, t)}{\partial t} = \frac{\partial X(y, t)}{\partial t} + \frac{\partial X}{\partial y} \frac{\partial y}{\partial t} \quad (23)$$

the dimensionless system of equations takes the following form:

$$\frac{1}{y^2} \frac{\partial}{\partial y} \left[y^2 \bar{k}_g(\bar{T}) \frac{\partial \bar{T}_g}{\partial y} \right] = C_2 4\pi \frac{\bar{W}}{\bar{R}} \bar{H}(y; \bar{R}), \quad (24)$$

$$\frac{1}{\lambda^2} \frac{\partial \bar{T}}{\partial \tau} = \frac{1}{y^2 \bar{R}^2} \frac{\partial}{\partial y} \left[y^2 \bar{k}(\bar{T}) \frac{\partial \bar{T}}{\partial y} \right] - \frac{1}{\lambda^2 \bar{R} y^2} \frac{d\bar{R}}{d\tau} \frac{\partial \bar{T}}{\partial y} + \frac{y}{\lambda^2 \bar{R}} \frac{d\bar{R}}{d\tau} \frac{\partial \bar{T}}{\partial y}, \quad (25)$$

$$\frac{\partial \bar{c}}{\partial \tau} = \frac{1}{y^2 \bar{R}^2} \frac{\partial}{\partial y} \left[y^2 \bar{D}(\bar{T}) \frac{\partial \bar{c}}{\partial y} \right] - \frac{1}{\bar{R} y^2} \frac{d\bar{R}}{d\tau} \frac{\partial \bar{c}}{\partial y} + \frac{y}{\bar{R}} \frac{d\bar{R}}{d\tau} \frac{\partial \bar{c}}{\partial y}, \quad (26)$$

$$\bar{W}(\tau) = \bar{R}^2 \left(C_1 h \frac{d\bar{R}}{d\tau} - \frac{\bar{k}(1)}{\bar{R}} \left(\frac{\partial \bar{T}}{\partial y} \right)_{y=1} \right), \quad (27)$$

$$\frac{d\bar{R}}{d\tau} = -\frac{F}{h} \bar{D}(1) \frac{1}{\bar{R}} \left(\frac{\partial \bar{c}}{\partial y} \right)_{y=1} = \frac{P}{\bar{R}}, \quad (28)$$

$$h = 3 \int_0^1 \frac{y^2}{1 + \bar{T}_g} dy. \quad (29)$$

Substitution of Eq. (28) in Eqs. (25) and (26) and multiplication of both with \bar{R}^2 brings them to a form in which the left-hand side depends explicitly only on time and the right-hand side depends explicitly only on y . The notation $\bar{H}(y; R)$ means that the heat source distribution may have an explicit dependence on bubble radius (and consequently on time) in addition to its y dependence. Actually, this is the situation from a practical point of view, i.e., a spherical heat source of finite size.

There are two specific cases in which this additional dependence is absent, i.e., $\bar{H} = \bar{H}(y)$ and a self-similarity solution of the problem is possible. The first case refers to a *uniform source* according to which the energy is evenly produced over the whole volume of the bubble ($\bar{H} = 3/4\pi$) and the second case is a *point source* at the center of the bubble ($\bar{H} = \delta(y)/4\pi$ where δ is the Dirac function). This second case is a good representation model for heating with a thermistor in the center of the bubble as far as the radius of the bubble is much larger than that of the thermistor. The point heat source constitutes the upper limit as regards the influence of the intrabubble heat transfer to the bubble growth rate. The other limit is that of heat generation at the bubble surface, leading to a uniform temperature in the bubble irrespective of the other parameters. Combining Eqs. (27) and (28) one can find that the ratio

$$\frac{\bar{W}(\tau)}{\bar{R}(\tau)} = C_1 h P - \bar{k}(1) \left(\frac{\partial \bar{T}}{\partial y} \right)_{y=1} = Z \quad (30)$$

does not depend explicitly on time. So, Eq. (24) with a point or uniform heat source can be solved for the self-similar temperature distribution in the bubble, $T_g(y)$. According to Eq. (30) a requirement for the existence of the self-similar bubble growth is that the external heating rate of the bubble must be proportional to its radius evolution.

When the bubble evolves in a self-similar fashion, the time derivatives of the transformed concentration and temperature

in the liquid can be set equal to zero so their spatial profiles can be found by solving the following equations:

$$\frac{1}{y^2} \frac{\partial}{\partial y} \left[y^2 \bar{D}(\bar{T}) \frac{\partial \bar{c}}{\partial y} \right] - \left(\frac{1}{y^2} - y \right) P \frac{\partial \bar{c}}{\partial y} = 0, \quad (31)$$

$$\frac{1}{y^2} \frac{\partial}{\partial y} \left[y^2 \bar{k}(\bar{T}) \frac{\partial \bar{T}}{\partial y} \right] - \left(\frac{1}{y^2} - y \right) \frac{P}{\lambda^2} \frac{\partial \bar{T}}{\partial y} = 0 \quad (32)$$

in $1 < y < \infty$ with boundary conditions $\bar{T}(1) = \bar{c}(1) = 1$, $\bar{T}(\infty) = \bar{c}(\infty) = 0$.

The constant P must be chosen to fulfill the relationship.

$$P = -\frac{F}{h} \bar{D}(1) \left(\frac{\partial \bar{c}}{\partial y} \right)_{y=1}. \quad (33)$$

For a finite initial radius of the bubble, the above self-similar profiles are reached after just a short transient (since the initial conditions are $\bar{c}(y, 0) = \bar{T}(y, 0) = 0$, this transient lasts until the bubble radius becomes several times its initial value). As the initial bubble radius tends to zero the transient period is becoming infinitely small and eventually at the limit $R_0 = 0$ the above self-similar solution is valid at all times. For the case of zero initial size, Eq. (28) is integrated to give

$$\bar{R} = \sqrt{2P\tau}. \quad (34)$$

From Eq. (30) results that $\bar{W} = Z\sqrt{2P\tau}$, i.e., the heating rate of the bubble must be proportional to the square root of time. This is the condition for retaining constant temperature on the bubble surface. It is important to notice that our focusing to the particular problem (admitting self-similar solution) is not emerging from some experimental indication on the bubble heating rate evolution but from experimental indication on the constancy of bubbles temperature at least for some period of the growth process (Divinis et al., 2004).

The derivation of the self-similar problem (Eqs. (31)–(34)) in the present work is not the usual one (actually is more general). During the well-known self-similarity transformation of the parabolic PDE of heat transfer the two independent variables collapse to one and the two boundaries and one initial condition collapse to two boundary conditions for the new variable. Here the self-similarity transformation is used to immobilize the domain of the problem and to made it more easily amenable to numerical treatment. The initial condition for finite bubble radius precludes the collapse of the boundary condition so time is retained as the second independent variable of the transformed problem. As the time proceeds the size scale of the system (proportional to bubble radius) becomes larger and larger with respect to the initial size scale so the initial conditions influences less and less the process and can be ignored. The self-similar solution is the asymptotic large time limit (or equivalently large R) of the transient Eqs. (25) and (26). This has been shown by Vrentas et al. (1983). If R_0 is small enough then the self-similar solution is very close to the actual solution for any bubble size of practical interest. In other words the transient

period is the time needed by the system to from the actual initial condition (temperature and concentration fields) to the self-similar one.

The validity of the present approach is confirmed by the fact that the self-similar Eqs. (31)–(34) can be also derived in the conventional way (i.e., by assuming $\bar{R} = 2\beta\sqrt{\tau}$ and using the transformed variable $\eta = \frac{r}{2\sqrt{\tau}}$) followed by the simple replacements $P = 2\beta^2$ and $y = \frac{\eta}{\sqrt{P/2}}$.

4. Problem solution

4.1. Heat transfer problem

4.1.1. Solution

The heat transfer Eq. (32) depends on mass transfer only through the growth parameter P , so the heat transfer problem can be decoupled, using $A = P/\lambda^2$ as a parameter, in order to study the influence of a temperature-dependent conductivity of the liquid. The equation for the heat transfer profile is

$$\frac{\partial^2 \bar{T}}{\partial y^2} + \left[\frac{1}{\bar{k}(\bar{T})} \frac{\partial \bar{k}(\bar{T})}{\partial \bar{T}} \frac{\partial \bar{T}}{\partial y} + \frac{2}{y} - \frac{1}{\bar{k}(\bar{T})} \left(\frac{1}{y^2} - y \right) A \right] \times \frac{\partial \bar{T}}{\partial y} = 0. \quad (35)$$

The liquid conductivity is usually a linear function of temperature at the temperature range of interest (from T_0 to T_e). If this is not the case, approximating conductivity with a linear function is usually a good choice. So it is assumed that $\bar{k}(\bar{T}) = 1 + \alpha\bar{T}$. For the case of a constant conductivity ($\alpha = 0$), the above equation has the following solution:

$$\bar{T} = \frac{\int_y^\infty x^{-2} e^{-(x^{-1} + x^2/2)A} dx}{\int_1^\infty x^{-2} e^{-(x^{-1} + x^2/2)A} dx}. \quad (36)$$

The parameter A is the ratio between convective (due to bubble expansion) and conductive contribution to heat transfer rate. In the limit $A \rightarrow 0$ the conduction dominates and a quasi-steady state temperature profile (with a length of several bubble radius) is developed. In the limit of very large A , the conduction is very slow in comparison with the interface movement and a very thin temperature profile is developed around the bubble.

Eq. (35) with the corresponding boundary conditions constitutes a boundary value problem. The shooting method is necessary for the solution of this problem because the large range of the extent of the temperature profile makes the use of finite differences inappropriate. This method has been also employed by Cable and Frade (1987b) for self-similar bubble growth with concentration-dependent diffusivity. First, the temperature and the temperature gradient are assumed to be 1 and 0, respectively, at a large y value. Using these values as initials, Eq. (35) is integrated backwards down to the bubble surface using an explicit Runge–Kutta integrator with variable step size (Press et al., 1986). If $\bar{T}(1) \neq 1$, the

Table 1

Exact and approximate values of the ratio f (heat flux for $\bar{k}(\bar{T})=1+0.5\bar{T}$ to heat flux for $\bar{k}(\bar{T})=1$) for several values of the parameter A

A	f exact	f approximate
0.01	1.244	1.235
0.1	1.228	1.211
0.2	1.220	1.201
0.5	1.207	1.185
1	1.197	1.173
2	1.187	1.162
5	1.175	1.149
10	1.168	1.141
50	1.157	1.129

initial temperature value is corrected appropriately and the procedure is repeated until the bubble surface temperature converges to 1.

4.1.2. Effect of variable conductivity

Lets define as f the ratio of the heat flux at the bubble surface between the cases of variable and constant conductivity computed at T_0 . It is well known that for the pure conductive ($A = 0$) case:

$$f = \int_0^1 \bar{k}(\bar{T}) d\bar{T} = k_m. \quad (37)$$

In Table 1 the computed values of f for several values of A and for $\alpha=0.5$ are shown. This value of α is considered to be an upper limit of the conductivity variation under realistic conditions. An approximate computation of f based on the constant conductivity solution but with A/k_m instead of A (rescaling of A since it includes conductivity through λ^2) and a multiplication with k_m , i.e.,

$$-\bar{k}(1) \left(\frac{\partial \bar{T}}{\partial y} \right)_{y=1} = \frac{k_m e^{-3A/2k_m}}{\int_1^\infty x^{-2} e^{-(x^{-1} + (x^2/2)A/k_m)A} dx} \quad (38)$$

is also performed and the results are shown in Table 1. Obviously, the above approximation is very successful and its error is around 2% even though the variation of conductivity is about 50%.

The above approximation cannot be used for larger and non-linear diffusivity variations. For example, for an exponential dependence of the form $\bar{k}(\bar{T}) = e^{-3\bar{T}}$ (similar to the diffusivity–concentration dependence studied by Lastochkin and Favelukis, 1998) which implies a 20-fold variation of the conductivity, the approximate solution reaches a 25% error as A goes to infinity.

As regards the effect of the variable conductivity on the temperature profile, the exact and approximate temperature profiles for very slow ($A \ll 1$, conduction dominated heat transfer) and very fast ($A \gg 1$, convection dominated heat transfer) bubble expansion are shown in Fig. 1. The scale in the horizontal axes is such to allow all curves to be displayed in one figure. In the case of fast growth, the approximate profile for $\bar{k} = 1$ (use of the far-field conductivity) and $\bar{k} = k_m$

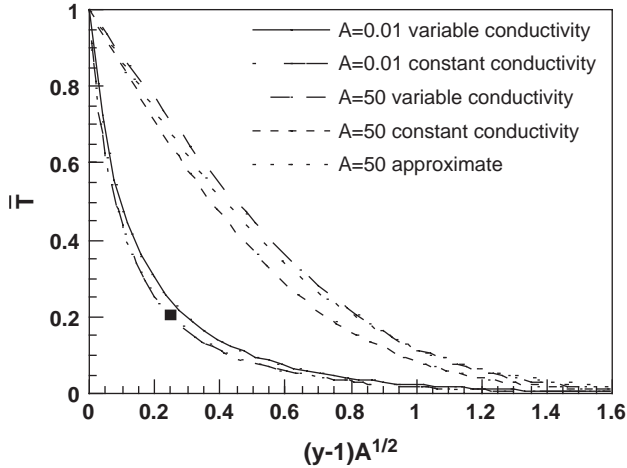


Fig. 1. Dimensionless temperature profile in the liquid for several forms of the thermal conductivity and several values of the parameter A .

(use of the mean conductivity) are shown separately. The above two approximations coincide when the growth is slow. Apparently, in all cases the difference between the exact and approximate temperatures is small. For a reasonable dependence of the diffusivity on temperature, the effect of using an approximate temperature profile on the diffusivity is inferred to be very small.

4.2. Mass transfer problem

4.2.1. Solution

The equation which must be solved for the concentration distribution is

$$\frac{\partial^2 \bar{c}}{\partial y^2} + \left[\frac{1}{\bar{D}(\bar{T})} \frac{\partial \bar{D}(\bar{T})}{\partial y} + \frac{2}{y} - \frac{1}{\bar{D}(\bar{T})} \left(\frac{1}{y^2} - y \right) P \right] \times \frac{\partial \bar{c}}{\partial y} = 0. \quad (39)$$

A first integration of the above equation leads to

$$\frac{\partial \bar{c}}{\partial y} = \text{constant} \times \exp \left[\int \left(-\frac{1}{\bar{D}(\bar{T})} \frac{\partial \bar{D}(\bar{T})}{\partial y} - \frac{2}{y} + \frac{P}{\bar{D}(\bar{T})} \left(\frac{1}{y^2} - y \right) \right) dy \right]. \quad (40)$$

Performing a second integration and using the condition $\bar{c}_1 = 1$, the concentration profile is found as

$$\bar{c} = \int_y^\infty \frac{1}{x^2 \bar{D}(\bar{T})} e^{P \int_1^x (z^{-2} - z) / \bar{D}(\bar{T}) dz} dx \times \left(\int_1^\infty \frac{1}{x^2 \bar{D}(\bar{T})} e^{P \int_1^x (z^{-2} - z) / \bar{D}(\bar{T}) dz} dx \right)^{-1}. \quad (41)$$

Substitution of this in Eq. (33) leads to

$$P = \frac{F}{h} \left(\int_1^\infty \frac{1}{x^2 \bar{D}(\bar{T})} e^{P \int_1^x (z^{-2} - z) / \bar{D}(\bar{T}) dz} dx \right)^{-1}. \quad (42)$$

The reduced density h depends on the temperature distribution inside the bubble through relation (29). This distribution can be found by solving Eq. (24). Using the Kirchoff transformation it can be shown that its general solution has the form

$$\int_0^{\bar{T}_g} \bar{k}_g(T) dT = ZC_2 \Psi(y), \quad (43)$$

where the function Ψ depends on the form of the source function H . In order to get some explicit results, a linear dependence between the gas conductivity and the temperature is assumed ($\bar{k}_g = 1 + E\bar{T}_g$). In general, this is not the case (Reid et al., 1986) but it is always a good approximation for small temperature variations. The temperature distribution for the cases of *point* and *uniform* sources are found as

$$\bar{T}_g = -E^{-1} + (E^{-2} + 2E^{-1} ZC_2 \Psi(y)), \quad (44)$$

where $\Psi(y) = \frac{1}{y} - 1$ for a point source and $\Psi(y) = \frac{1-y^2}{2}$ for a uniform source. Substituting Eq. (44) in Eq. (29), a non-linear equation with respect to h (since h is included in Z) is obtained.

For the special case of a constant conductivity in the bubble ($E = 0$), the following closed form equations must be solved for h :

$$h/3 = \frac{1 - (C_2 Z)^3}{3(1 - C_2 Z)^4} - \frac{3C_2 Z(1 - (C_2 Z)^2)}{2(1 - C_2 Z)^4} + \frac{3(C_2 Z)^2}{(1 - C_2 Z)^3} + \frac{(C_2 Z)^3}{(1 - C_2 Z)^4} \ln(C_2 Z), \quad (45a)$$

$$h/3 = \frac{-2}{C_2 Z} \left[1 + \frac{1}{2} \left(1 + \frac{2}{C_2 Z} \right)^{1/2} \times \ln \left(\frac{(1 + (2/C_2 Z))^{1/2} - 1}{(1 + (2/C_2 Z))^{1/2} + 1} \right) \right] \quad (45b)$$

for the point and uniform source, respectively.

Eq. (42) must be solved for P . The solution procedure is the following: a value of P is chosen (outer loop). The heat transfer boundary value problem is solved (inner loop), as has already been shown, for the temperature profile. This temperature profile is used next for the numerical computation of the integrals in Eq. (42). A new value of P is computed then and the above steps are repeated until P converges to a certain value.

Although the above procedure is simple, its convergence (especially for large values of P) is difficult. An alternative approach to find P , is to solve directly the boundary value problem consisting of Eqs. (35) and (39). The following shooting method is employed for this purpose: values for the temperature and concentration gradients at $y = 1$ and

a value of P are assumed. The system of Eqs. (35) and (39) is integrated forward numerically with initial values $\bar{T}(1) = \bar{c}(1) = 1$ and the assumed gradient values until the temperature and concentration become independent of y . The values of \bar{T} , \bar{c} at this point must be zero. The initial gradients and the estimation of P are corrected using the Newton Raphson method in order to make the asymptotic temperature and concentration values zero and to fulfill Eq. (33). The above procedure is repeated until convergence is achieved.

4.2.2. Effect of variable diffusivity

A first approximation is to use the following approximate temperature profile and temperature flux in Eqs. (41) and (42) instead of solving the thermal boundary value problem:

$$\bar{T} = \frac{\int_1^\infty x^{-2} e^{-(x^{-1}+x^2/2)P/\lambda^2 k_m} dx}{\int_1^\infty x^{-2} e^{-(x^{-1}+x^2/2)P/\lambda^2 k_m} dx}. \quad (46)$$

This has the advantage that Eqs. (41) and (42) can be written in a closed form and only one iteration level (for P) is needed.

Lets assume (at least as a first order approximation) that the relation between diffusivity and temperature is linear in the range of interest, i.e., $\bar{D}(\bar{T}) = 1 + \gamma\bar{T}$. A typical range of values for γ is from zero to 0.5. The exact values of P and those computed using Eq. (46) differ from each other by less than 1% for $\alpha = \gamma = 0.5$, $\lambda = 1$ and several values of F . This means that the approximate temperature profile can be used safely for this degree of variation of conductivity and diffusivity.

A further approximation relies on the assumption of a constant diffusivity similar to what was done for the conductivity. By using a constant diffusivity D_m in Eq. (42), it can be shown that $P = D_m \bar{P}$ where the parameter \bar{P} can be found from the solution of the following transcendental equation:

$$\bar{P} = F e^{-3\bar{P}/2} \int_1^\infty \frac{1}{x^2} e^{-\bar{P}(x^{-1}+x^2/2)} dx. \quad (47)$$

The appropriate value of the constant diffusivity can be computed by averaging the actual value of diffusivity over the liquid weighted by an approximate concentration gradient. To avoid iterative procedures, the values of the convection parameter P needed for the computation of concentration and temperature profiles are assumed to be $D_c \bar{P}$ and $D_T \bar{P}$, respectively. The values D_c and D_T must be chosen to fit the exact results.

Finally, to make the approximation simpler, Eq. (47) is solved using a combination of the Newton Raphson method with a numerical integration to find the value of \bar{P} for several values of the parameter F . The results can be fitted using elementary functions with an accuracy better than 1%. Collecting all the above steps leads to the following explicit

approximate solution for the growth parameter P :

$$P_{\text{apr}} = D_m \bar{P}, \quad (48a)$$

$$D_m = \left(\int_1^\infty \frac{1}{x^2 D(\bar{T})} e^{-D_c \bar{P}(x^{-1}+x^2/2)} dx \right)^{-1} \times \int_1^\infty \frac{1}{x^2} e^{-D_c \bar{P}(x^{-1}+x^2/2)} dx, \quad (48b)$$

$$\bar{T} = \frac{\int_1^\infty x^{-2} e^{-(x^{-1}+x^2/2)D_T \bar{P}/\lambda^2 k_m} dx}{\int_1^\infty x^{-2} e^{-(x^{-1}+x^2/2)D_T \bar{P}/\lambda^2 k_m} dx}, \quad (48c)$$

$$\bar{P} = 2(0.7063 + 0.4545\sqrt{F} + 0.1591F)^2 F \quad \text{for } F < 0.89,$$

$$\bar{P} = 2(F + 0.3232)^2 \quad \text{for } 0.89 < F < 5.7,$$

$$\bar{P} = 2(F\sqrt{3/\pi} + 4/9)^2 \quad \text{for } F > 5.7. \quad (49)$$

It is found that the best choices for the parameters D_c and D_T is $\bar{D}(1)$ and 1, respectively. The distinct advantage of the approximate solution over the exact is that no iterations are needed.

Actually, \bar{P} is the growth parameter for the case of constant diffusivity equal to its far-field value (i.e., $\bar{D}(1) = 1$). The effect of a varying diffusivity can be assigned to a correction factor C_f such that $P = C_f \bar{P}$. In this respect, D_m is an approximation to C_f . The correction factor takes values between 1 for a temperature profile much thinner than the concentration profile (i.e., $\lambda \ll 1$) and $\bar{D}(1)$ for the opposite case (i.e., $\lambda \gg 1$). For a given temperature dependence of conductivity and diffusivity the correction factor is a function of the Foaming number F and the ratio λ .

The correction factor for a constant conductivity and a linear diffusivity with $\gamma = 0.5$ and for three values of F , is plotted against λ in Fig. 2. At the limit $F = 0$, the factor C_f takes the constant value $0.5/\ln(1.5)$ irrespective of the λ value. At the region of $\lambda = 1$, the temperature and concentration profiles are comparable so the correction factor takes

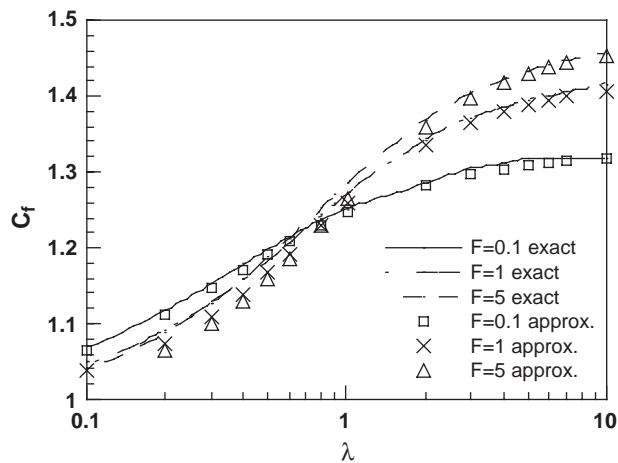


Fig. 2. Exact and approximate values of the correction factor C_f versus the parameter λ for several values of the Foaming number F .

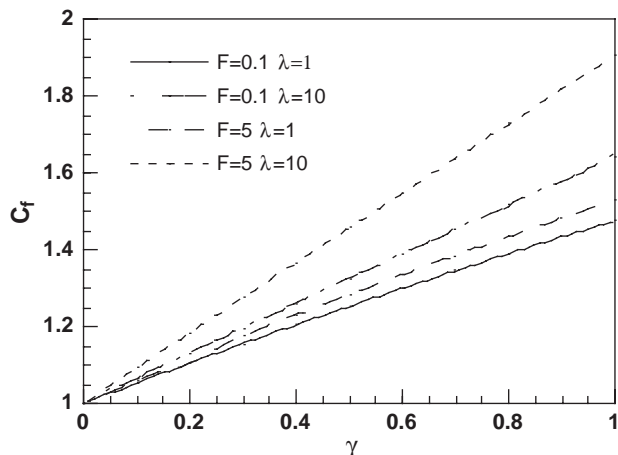


Fig. 3. Dependence of the correction factor C_f on the slope β of the diffusivity–temperature dependence ($\bar{D}(\bar{T}) = 1 + \gamma\bar{T}$) for several pairs of the parameter λ and the Foaming number F .

values in the region of 1.25 which is the average between 1 and $\bar{D}(1)$.

In all cases, C_f is an increasing function of λ . For a slow growth rate (small F), C_f reaches a maximum which is smaller than $\bar{D}(1)$. This is due to the fact that the concentration profile is already thick so there is always an interaction with the temperature profile even if the latter is in a pseudo-steady state. The $\lambda < 1$ section of the curve $C_f(\lambda)$ converges fast to a particular shape as the value of F increases. The upper limit $\bar{D}(1)$ at $\lambda \rightarrow \infty$ is accessible only in the case of fast growth. The agreement between the exact and approximate correction factors is fair. Significant deviations are observed for $\lambda \ll 1$. However, it must be noted that from a practical point of view only the region $\lambda > 1$ is of interest since typically heat diffusivity in liquids is larger than mass diffusivity of gases in liquids. The approximate solution performs very well in the above region and can be safely used.

The dependence of the correction factor C_f on the diffusivity variation parameter γ is shown in Fig. 3 for two cases of a small and a large growth rate ($F = 0.1$ and 5, respectively) and two values of λ . It must be noted that for the pseudo-steady state growth ($F = 0$), the above dependence can be found in closed form as $C_f = \gamma / \ln(1 + \gamma)$. In all cases, the curves up to 1 can be fitted exactly (machine accuracy) with a parabola. An approximation with a straight line also performs very well (variation coefficient more than 0.999) and can be used with confidence. This means that the approximate correction factor (48b) must be computed once (for just a single value of β) to give the slope of the C_f versus γ . After this, the computation of C_f for arbitrary value of β (diffusivity–temperature dependence) is trivial.

The exact and approximate (assuming a constant diffusivity $\bar{D} = D_m$) concentration profiles for the cases of slow ($F = 0.01$) and fast ($F = 5$) bubble growth, are shown in Fig. 4. The x -axis has been scaled properly to allow pre-

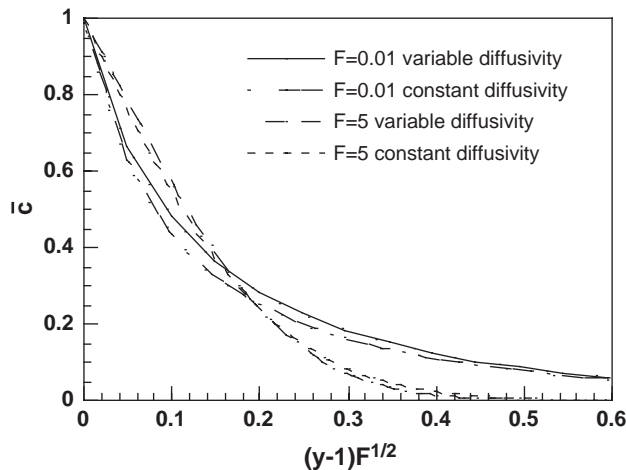


Fig. 4. Dimensionless concentration profile in the liquid for variable ($\bar{D}(\bar{T}) = 1 + 0.5\bar{T}$) and constant ($\bar{D}(\bar{T}) = D_m$) diffusivity and for two values of the Foaming number F .

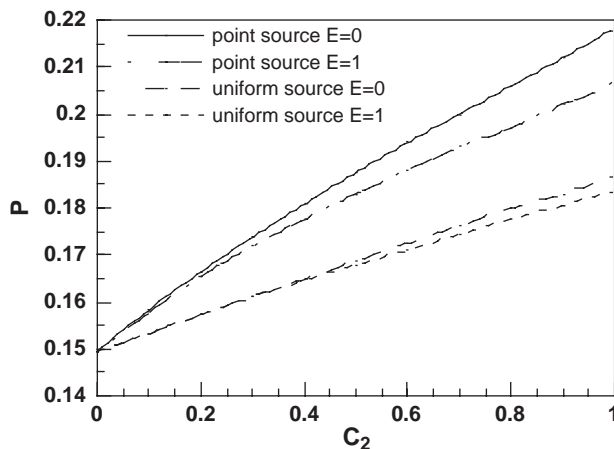


Fig. 5. Bubble growth parameter P versus parameter C_2 for the cases of point and uniform heat sources and two values of the gas conductivity–temperature dependence parameter E ($F = 0.1$, $\lambda = 5$).

sentation of both profiles in the same scale. The deviation between exact and approximate profiles is getting smaller as F increases. In general, the approximation of the profiles is better than the approximation of the surface flux since the latter includes the derivative of the profile (so it requires a higher order approximation).

4.2.3. Effect of the intrabubble temperature distribution

Although the assumption of $C_1 \approx 0$ holds almost in all practical cases, the parameter C_2 can be of the order $O(1)$ leading to a rise of the intrabubble temperature concomitant with the reduction of its density and increase of the growth rate. The influence of the parameter C_2 on the growth rate constant P for values of C_2 between 0 (uniform temperature in the bubble) and 1 is shown in Fig. 5, for the cases of slow growth ($F = 0.1$) with $\lambda = 5$. In general, P in-

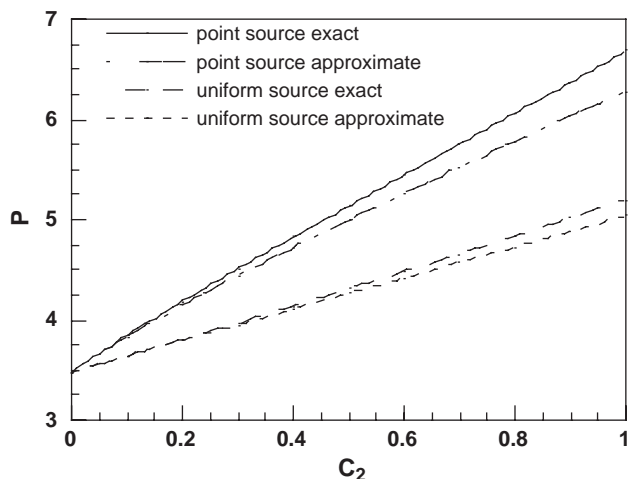


Fig. 6. Exact and approximate values of bubble growth parameter P versus parameter C_2 for point and uniform heat sources in the bubble ($F = 1$, $\lambda = 5$).

creases with C_2 . This increase is larger for the case of the point source than the case of uniform source since in the former, the non-uniformity of the temperature field in the bubble is more pronounced. This increase is also larger for the case of a constant gas conductivity ($E = 0$) than for a temperature-dependent one ($E = 1$). The increase of the conductivity due to a local temperature rising leads to more uniform temperature distributions and to reduced growth rates.

The existence of a non-zero C_2 makes necessary the use of an iterative technique for solving Eq. (42) even for the case of constant transport properties. An approximate procedure that can be used is (i) estimation of P from Eq. (49) (ii) use of the previous value of P to compute a temperature gradient (Eq. (38)) and h (Eqs. (29), (30) and (44)) (iii) Use of a modifying Foaming number F/h to compute P from Eq. (49). The exact and approximate values of P are shown in Fig. 6 versus the parameter C_2 for $F = 1$ and $\lambda = 5$. Evidently, as the growth gets faster (larger F) the influence of C_2 on P increases. The approximate technique is appropriate for $C_2 < 0.5$. The approximate techniques for taking into account the transport properties, temperature dependence and the intrabubble temperature distribution can be combined by correcting first for the variable properties (Eqs. (48) and (49)) and using the value P_{apr} to evaluate the normalized density h and the final estimation for P .

5. Concluding remarks

The problem of bubble growth of a desorbing gas inside a liquid, due to an internal heat source in the bubble, has been considered in this work. It is shown that, under certain conditions for the heat source, the bubble grows in a self-similar manner even for the cases of temperature-dependent properties and non-uniform temperature inside the bubble.

The bubble radius grows proportional to the square root of time with a proportionality constant, which can be found from the solution of a boundary value problem consisting of two second order ordinary differential equations in the semi-infinite domain. Several approximating procedures to find this proportionality constant are derived and assessed. The influence of the temperature dependence of the transport properties on the bubble growth rate is examined in detail.

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